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A study of the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$ *

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Abstract In this paper, we introduce the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$ and we investigate its limit formulas, integral representations, differentiation formulas, infinite sums, recursion formulas, the Laplace, Mellin, fractional Fourier, double Laplace and double Mellin transforms for the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$. Finally, we discuss the fractional integration and the k-fractional differentiation .

Key words Horn's hypergeometric $\mathbf{H}_9$, the k- Pochhammer symbol, limit formulas, recursion formulas, Upadhyaya transform, Laplace transform, Mellin transform, fractional Fourier transform, double Upadhyaya transform, double Laplace transform, double Mellin transform.

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1 Introduction and Preliminaries

Recently, many functions are expanded by adding the k-parameter to their definitions using the k-gamma function and the k- Pochhammer symbol, for example, see [11–14]. Our present attempt is one more step in that direction.
The paper is organized as follows. In subsection 1.1, we recall some important definitions which are used in this paper. In Section 2 we define the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$, with the enunciation of the limit formulas for this function in Section 3. In Section 4 we establish the integral representations for the introduced function. In Sections 5 and 6 we derive the differentiation formulas and the infinite sums respectively. The recursion formulas are presented in Section 7, while the Laplace, Mellin and fractional Fourier transforms for this function are explored in Section 8. The double Laplace and the double Mellin transforms of the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$ are established in Section 9. Section 10 is devoted to the formulas of fractional integration and k-fractional differentiation of this function. The concluding remarks in the last section finish the paper.

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1.1 Preliminaries

In this subsection below we recall some important definitions which are used in this paper.

Definition 1.1. For $x \in \mathbb{C}$,$k \in \mathbb{R}^+$ and $\Re(x) > 0$, the integral representation of k-gamma function is defined as (see [3,8]),

$$\begin{aligned} \Gamma_k(x) &= \int_0^\infty e^{-\frac{t^k}{k}} t^{x-1} dt, \\ \Gamma_k(x+k) &= x\Gamma_k(x), \\ \Gamma_k(x) &= k^{\frac{x}{k}-1}\, \Gamma(\frac{x}{k}). \end{aligned} \tag{1.1}$$

Definition 1.2. For $x, y \in \mathbb{C}$,$k \in \mathbb{R}^+$, $\Re(x) > 0$ and $\Re(y) > 0$, the integral representation of the k-beta function is defined as (see [3]),

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}\, (1-t)^{\frac{y}{k}-1} dt, \tag{1.2}$$

The relation between the k-beta and the k-gamma function is

$$B_k(x, y) = \frac{\Gamma_k(x)\, \Gamma_k(y)}{\Gamma_k(x+y)}$$

Definition 1.3. Let $x \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $m \in \mathbb{N}$, the k-Pochhammer symbol is defined as ([3,8]),

$$\begin{aligned} (x)_{m,k} &= \frac{\Gamma_k(x+mk)}{\Gamma_k(x)} = \begin{cases} x(x+k)(x+2k)\ldots(x+(m-1)k), & m \in \mathbb{N}; \\ 1, & m = 0 \end{cases}, \\ (x)_{m-n,k} &= (x)_{m,k}(x+mk)_{-n,k}, \\ (x)_{-m,k} &= \frac{(-1)^{-mk}}{(k-x)_{m,k}}, \\ (x)_{2m,k} &= 2^{2m}\, (\frac{x}{2})_{m,k}\, (\frac{x+k}{2})_{m,k}, \\ (x)_{m-n+1,k} &= x(x+1)_{m-n,k}, \\ (x)_{m-n-1,k} &= \frac{1}{x-1}(x-1)_{m-n,k}. \end{aligned} \tag{1.3}$$

Definition 1.4. The Laplace transform of $f(z)$, which is a particular case of the Upadhyaya transfrom (see, Upadhyaya [17], Upadhyaya et al. [18]), is defined as (see, [7])

$$\pounds\{f(z) : s\} = \int_0^\infty e^{-sz} f(z)dz, \Re(s) > 0. \tag{1.4}$$

Definition 1.5. The Mellin transform of $f(z)$ is defined as (see, [6])

$$M\{f(z) : s\} = \int_0^\infty z^{s-1} f(z)dz. \tag{1.5}$$

Definition 1.6. Let f be a function belonging to the Lizorkin space $\Phi(R)$ then the fractional Fourier transform of $f(x)$ of order α is defined as (see, [4])

$$\Im_\alpha[f(x) : \omega] = \int_R e^{i\omega^{\frac{1}{\alpha}} x} f(x)dx,\ \omega > 0, \tag{1.6}$$

where $0 < \alpha \leq 1$.

Definition 1.7. The double Laplace transform of a function (which is a special case of the double Upadhyaya transform (see, Upadhyaya [17], Upadhyaya et al. [18]), $f(x, y)$ of two variables x and y defined in the first quadrant of the xy-plane is defined as (see, [9])

$$\pounds_2\{\ f(x, y)\ \} = \int_0^\infty \int_0^\infty f(x, y)\, e^{-(rx+sy)} dxdy,\ \Re(r, s) > 0. \tag{1.7}$$

Definition 1.8. The double Mellin transform of a function $f(x,y)$ of two variables x and y is defined as (see, [10])

$$M_{xy}\{\ f(x,y):r,s\ \}=\int_0^\infty\int_0^\infty f(x,y)x^r y^s dxdy. \tag{1.8}$$

Definition 1.9. Let f be a sufficiently well behaved function which has its support in $\mathbb{R}^+$ and let $\alpha \succ 0$ be a real number. The k-Riemann-Liouville fractional integral is defined as (see, [15])

$$I_k^\alpha\ f(x)=\frac{1}{k\ \Gamma_k(\alpha)}\int_0^x\ (x-t)^{\frac{\alpha}{k}-1}\ f(t)\ dt,\quad x\in\mathbb{R}^+,\alpha\in\mathbb{C},\ \Re(\alpha)>0. \tag{1.9}$$

Definition 1.10. Let f be a sufficiently well behaved function which has its support in $\mathbb{R}^+$ and let $\alpha \succ 0$ be a real number. The k-Riemann-Liouville fractional derivative is defined as (see, [16])

$$D_k^\alpha\ f(x)=\frac{d}{dx}\ I_k^{1-\alpha}\ f(x). \tag{1.10}$$

2 The k-Horn's hypergeometric function $\mathbf{H}_{9,k}$

In this section we define the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$

Definition 2.1. For $k>0$, we define the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$ as follows:

$$\mathbf{H}_{9,k}\,(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}m!n!}x^m y^n;|x|<\frac{1}{4},|y|<\infty, \tag{2.1}$$

where $(\alpha)_{m,k}$ is the k-Pochhammer symbol defined in (1.3).

Special Case: For $k=1$, the result gives the conventional Horn's hypergeometric function H_9 defined in [2], [1, p. 226, (37)].

$$H_{9,1}(\alpha,\beta;\gamma;x,y)=H_9(\alpha,\beta;\gamma;x,y).$$

3 Limit formulas

In this section we present some limit formulas for the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$.

Theorem 3.1. *The limit formulas for $\mathbf{H}_{9,k}$ hold true:*

$$\lim_{\alpha\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x/\alpha^2\ ,\alpha y)=\ {}_0F_{1,k}\left[\begin{matrix}-\\(\gamma,k)\end{matrix}\ ;x\right]\cdot{}_1F_{0,k}\left[\begin{matrix}(\gamma,k)\\-\end{matrix}\ ;y\right], \tag{3.1}$$

$$\lim_{\beta\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x;\frac{y}{\beta})=\mathbf{H}_{10,k}(\alpha;\gamma;x,y), \tag{3.2}$$

where, (see also, Erdélyi et al. [1, (38), p. 226])

$$\mathbf{H}_{10,k}(\alpha;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}x^m y^n}{(\gamma)_{m,k}m!n!},\qquad |x|<\frac{1}{4},|y|<\infty,$$

$$\lim_{\gamma\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x;y)=\ {}_2F_{0,k}\left[\begin{matrix}\left(\frac{\alpha}{2},k\right),\left(\frac{\alpha+k}{2},k\right)\\-\end{matrix}\ ;4x\right]\cdot{}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(k-\alpha-2mk,k)\end{matrix}\ ;(-1)^{-k}\ y\right], \tag{3.3}$$

$$\lim_{\varepsilon\to0}\ \mathbf{H}_{9,k}\ (\frac{\alpha}{\varepsilon},\beta;\gamma;\varepsilon^2 x,y/\varepsilon)=\ {}_0F_{1,k}\left[\begin{matrix}-\\(\gamma,k)\end{matrix}\ ;\alpha^2 x\right]\cdot{}_1F_{0,k}\left[\begin{matrix}(\beta,k)\\-\end{matrix}\ ;y/\alpha\right], \tag{3.4}$$

$$\lim_{\varepsilon\to0}\ \mathbf{H}_{9,k}(\alpha,\frac{\beta}{\varepsilon};\gamma;x,\varepsilon y)=\ \mathbf{H}_{10,k}(\alpha;\gamma;x,\beta y), \tag{3.5}$$

$$\lim_{\varepsilon\to 0}\ \mathbf{H}_{9,k}(\alpha,\beta;\frac{\gamma}{\varepsilon};\frac{x}{\varepsilon},y)=\ {}_2F_{0,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\ -\end{array};\frac{4x}{\gamma}\right]\cdot{}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right], \tag{3.6}$$

$$\lim_{\varepsilon\to 0}\ \mathbf{H}_{9,k}(\alpha,\frac{\beta}{\varepsilon};\frac{\gamma}{\varepsilon};\frac{x}{\varepsilon},\varepsilon y)=\ {}_2F_{0,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\ -\end{array};\frac{4x}{\gamma}\right]\cdot{}_0F_{1,k}\left[\begin{array}{c}-\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ \beta y\right], \tag{3.7}$$

$$\lim_{\varepsilon\to 0}\ \mathbf{H}_{9,k}(\frac{\alpha}{\varepsilon},\frac{\beta}{\varepsilon};\gamma;\varepsilon^2\ x,y)=\exp(\beta y/\alpha)\ \cdot{}_0F_{1,k}\left[\begin{array}{c}-\\ (\gamma,k)\end{array};\alpha^2 x\right], \tag{3.8}$$

and

$$\lim_{\varepsilon\to 0}\ \mathbf{H}_{9,k}(\frac{\alpha}{\varepsilon},\frac{\beta}{\varepsilon};\frac{\gamma}{\varepsilon};\varepsilon\ x,y)=\exp(\beta y/\alpha)\ \cdot\exp(\alpha^2 x/\gamma). \tag{3.9}$$

Proof. We have

$$\lim_{\alpha\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x/\alpha^2\ ,\alpha y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_{n,k}\ x^m y^n}{(\gamma)_{m,k}\ m!n!}\cdot\lim_{\alpha\to\infty}\ \frac{(\alpha)_{2m-n,k}}{(\alpha)^{2m-n}},$$

Since we have

$$\lim_{\alpha\to\infty}\ \frac{(\alpha)_{2m-n,k}}{(\alpha)^{2m-n}}=\lim_{\alpha\to\infty}\ \frac{\alpha}{\alpha}\ \frac{(\alpha+k)}{\alpha}\ \frac{(\alpha+2k)}{\alpha}\cdots\ \frac{(\ \alpha+((2m-n)-1\ )k)}{\alpha}=1,$$

hence, we can write

$$\lim_{\alpha\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x/\alpha^2\ ,\alpha y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_{n,k}\ x^m y^n}{(\gamma)_{m,k}\ m!n!}=\ {}_0F_{1,k}\left[\begin{array}{c}-\\ (\gamma,k)\end{array};x\right]\cdot{}_1F_{0,k}\left[\begin{array}{c}(\beta,k)\\ -\end{array};y\right]$$

which the desired result (3.1).
Similarly, we write

$$\lim_{\beta\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x;\frac{y}{\beta})=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ x^m y^n}{(\gamma)_{m,k}m!n!}\cdot\lim_{\beta\to\infty}\ \frac{(\beta)_{n,k}}{(\beta)^n}=\mathbf{H}_{10,k}(\alpha,-;\gamma;x;y)=\mathbf{H}_{10,k}(\alpha;\gamma;x,y),$$

which produces (3.2),
and

$$\lim_{\gamma\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x;y)=\lim_{\gamma\to\infty}\ \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ (\gamma\ x)^m y^n}{(\gamma)_{m,k}m!n!}$$

$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ x^m y^n}{m!n!}\cdot\lim_{\gamma\to\infty}\ \frac{(\gamma)^m}{(\gamma)_{m,k}}=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ x^m y^n}{m!n!},$$

Since we have

$$(\alpha)_{2m-n,k}=(\alpha)_{2m,k}(\alpha+2mk)_{-n,k}=\frac{2^{2m}\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}}{(k-\alpha-2mk)_{n,k}},$$

hence, we obtain

$$\lim_{\gamma\to\infty}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x;y)=\sum_{m,n=0}^{\infty}\frac{4^m\ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}\ (\beta)_{n,k}\ x^m y^n}{(k-\alpha-2mk)_{n,k}\ m!n!}$$

$$=\ {}_2F_{0,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\ -\end{array};4x\right]\cdot{}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right],$$

which gives (3.3).
Also, we can write

$$\lim_{\varepsilon\to 0} \mathbf{H}_{9,k}(\frac{\alpha}{\varepsilon},\beta;\gamma;\varepsilon^2 x, y/\varepsilon) = \lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\frac{\alpha}{\varepsilon})_{2m-n,k}\,(\beta)_{n,k}\,(\varepsilon^2 x)^m(\frac{y}{\varepsilon})^n}{(\gamma)_{m,k}\;m!n!} = \lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\beta)_{n,k}\,x^m y^n}{(\gamma)_{m,k}\;m!n!}\cdot\frac{(\frac{\alpha}{\varepsilon})_{2m-n,k}}{(\frac{\alpha}{\alpha\varepsilon})^{2m-n}}$$

$$= \sum_{m,n=0}^{\infty}\frac{(\beta)_{n,k}(\alpha^2 x)^m(\frac{y}{\alpha})^n}{(\gamma)_{m,k}\;m!n!}\cdot\lim_{\varepsilon\to 0}\frac{(\frac{\alpha}{\varepsilon})_{2m-n,k}}{(\frac{\alpha}{\varepsilon})^{2m-n}} = {}_0F_{1,k}\left[\begin{matrix} - \\ (\gamma,k)\end{matrix};\alpha^2 x\right]\cdot {}_1F_{0,k}\left[\begin{matrix}(\beta,k)\\ -\end{matrix};y/\alpha\right],$$

thus proving (3.4).
Similarly, we can write

$$\lim_{\varepsilon\to 0} \mathbf{H}_{9,k}(\alpha,\frac{\beta}{\varepsilon};\gamma;x,\varepsilon y) = \lim_{\varepsilon\to 0}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,(\frac{\beta}{\varepsilon})_{n,k}\,x^m(\varepsilon y)^n}{(\gamma)_{m,k}\;m!n!} = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,x^m\,y^n}{(\gamma)_{m,k}\;m!n!}\cdot\lim_{\varepsilon\to 0}\frac{(\frac{\beta}{\varepsilon})_{n,k}}{(\frac{\beta}{\beta\,\varepsilon})^n}$$

$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,x^m\,(\beta\,y)^n}{(\gamma)_{m,k}\;m!n!}\cdot\lim_{\varepsilon\to 0}\frac{(\frac{\beta}{\varepsilon})_{n,k}}{(\frac{\beta}{\varepsilon})^n} = \mathbf{H}_{10,k}(\alpha;\gamma;x,\beta y),$$

thereby giving (3.5). Using the same technique we can prove (3.6), (3.7), (3.8) and (3.9) in a similar fashion. □

Theorem 3.2. *The following limit formulas for $\mathbf{H}_{9,k}$ hold good:*

$$\lim_{\alpha\to\infty}\lim_{\beta\to\infty}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x/\alpha^2,\alpha y/\beta) = \exp(y)\cdot {}_0F_{1,k}\left[\begin{matrix} - \\ (\gamma,k)\end{matrix};x\right], \tag{3.10}$$

$$\lim_{\alpha\to\infty}\lim_{\gamma\to\infty}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x/\alpha^2,\alpha y) = \exp(x)\cdot {}_1F_{0,k}\left[\begin{matrix}(\beta,k)\\ -\end{matrix};y\right], \tag{3.11}$$

$$\begin{aligned}&\lim_{\beta\to\infty}\lim_{\gamma\to\infty}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x,y/\beta)\\ &= {}_2F_{0,k}\left[\begin{matrix}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\ -\end{matrix};4x\right]\cdot {}_0F_{1,k}\left[\begin{matrix} - \\ (k-\alpha-2mk,k)\end{matrix};(-1)^{-k}\,y\right],\end{aligned} \tag{3.12}$$

and

$$\lim_{\alpha\to\infty}\lim_{\beta\to\infty}\lim_{\gamma\to\infty}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\gamma x/\alpha^2,\alpha y/\beta) = \exp(x)\cdot\exp(y). \tag{3.13}$$

Proof. The proof of this theorem is parallel to that of the Theorem 3.1. □

4 Integral representations

In this section,we establish several integral representations of the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$.

Theorem 4.1. *The following integral representations of the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$ hold true:*

$$\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y) = \frac{k^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\int_0^{\infty} t^{\frac{\alpha}{k}-1}\exp(-t)\;{}_0F_{1,k}\left[\begin{matrix} - \\ (\gamma,k)\end{matrix};k^2\,t^2\,x\right]\cdot {}_1F_{0,k}\left[\begin{matrix}(\beta,k)\\ -\end{matrix};y/kt\right]dt, \tag{4.1}$$

$$\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y) = \frac{k^{\frac{\beta}{k}-1}}{\Gamma_k(\beta)}\int_0^{\infty} t^{\frac{\beta}{k}-1}\exp(-t)\,\mathbf{H}_{10,k}(\alpha;\gamma;x,k\,t\,y)\,dt, \tag{4.2}$$

$$\begin{aligned}&\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\\ &= \frac{k^{\frac{\beta}{k}-1}}{\Gamma_k(\beta)}\int_0^{\infty} t^{\frac{\beta}{k}-1}\exp(-t)\;{}_2F_{1,k}\left[\begin{matrix}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\ (\gamma,k)\end{matrix};4x\right]\times\\ &{}_0F_{1,k}\left[\begin{matrix} - \\ (k-\alpha-2mk,k)\end{matrix};(-1)^{-k}\,k\,t\,y\right]dt,\end{aligned} \tag{4.3}$$

and

$$\int_0^\infty t^{s-1}\exp(-\rho t)\ \ {}_2F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\(\gamma,k)\end{array};4x\right]\times\ {}_0F_{1,k}\left[\begin{array}{c}-\\(k-\alpha-2mk,k)\end{array};(-1)^{-k}\ k\ t\ y\right]dt$$
$$=\frac{k\Gamma_k(ks)}{(k\rho)^s}\ \mathbf{H}_{9,k}(\alpha,ks;\gamma;x,y/\rho). \tag{4.4}$$

Proof. Using the integral formula of k-Pochhammer's symbol $(\alpha)_{2m-n,k}$ in the definition of the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$, we can write

$$\begin{aligned}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}x^my^n}{(\gamma)_{m,k}m!n!}\\&=\frac{1}{\Gamma_k(\alpha)}\int_0^\infty t^{\frac{\alpha}{k}+2m-n-1}\exp(-t)\ \sum_{m,n=0}^{\infty}\frac{(\beta)_{n,k}x^my^nk^{\frac{\alpha}{k}+2m-n-1}}{(\gamma)_{m,k}m!n!}\,dt\\&=\frac{k^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\int_0^\infty t^{\frac{\alpha}{k}-1}\exp(-t)\ {}_0F_{1,k}\left[\begin{array}{c}-\\(\gamma,k)\end{array};k^2\ t^2\ x\right]\cdot\ {}_1F_{0,k}\left[\begin{array}{c}(\beta,k)\\-\end{array};y/kt\right]dt,\end{aligned}$$

which is the required result of (4.1).
Also using the integral formula of the k- Pochhammer symbol $(\beta)_{n,k}$ in the definition of the k-Horn's hypergeometric function $\mathbf{H}_{9,k}$, we can write

$$\begin{aligned}\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)&=\frac{1}{\beta_k(\alpha)}\int_0^\infty t^{\frac{\beta}{k}+n-1}\exp(-t)\ \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,x^m\,y^n\,k^{\frac{\beta}{k}+n-1}}{(\gamma)_{m,k}m!n!}\,dt\\&=\frac{k^{\frac{\beta}{k}-1}}{\Gamma_k(\beta)}\int_0^\infty t^{\frac{\beta}{k}-1}\exp(-t)\ \mathbf{H}_{10,k}(\alpha;\gamma;x,k\ t\ y)\ dt.\end{aligned}$$

thereby proving (4.2). The proof of (4.3) is parallel to (4.2).
From (4.3), we can obtain the equality

$$\int_0^\infty t^{s-1}\exp(-\rho t)\ \ {}_2F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\(\gamma,k)\end{array};4x\right]\times$$
$${}_0F_{1,k}\left[\begin{array}{c}-\\(k-\alpha-2mk,k)\end{array};(-1)^{-k}\ k\ t\ y\right]dt=\frac{k\Gamma_k(ks)}{(k\rho)^s}\ \mathbf{H}_{9,k}(\alpha,ks;\gamma;x,y/\rho),$$

which can be regarded as the Laplace transform

$$\mathcal{L}\{f(t):\rho\}=\int_0^\infty e^{-\rho t}f(t)dt$$

or, the Mellin transform

$$M\{f(t):s\}=\int_0^\infty t^{s-1}f(t)dt$$

of the functions $t^{s-1}\ {}_2F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\(\gamma,k)\end{array};4x\right]\cdot\ {}_0F_{1,k}\left[\begin{array}{c}-\\(k-\alpha-2mk,k)\end{array};(-1)^{-k}\ k\ t\ y\right]$
and $\exp(-\rho t)\ {}_2F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k)\\(\gamma,k)\end{array};4x\right]\cdot\ {}_0F_{1,k}\left[\begin{array}{c}-\\(k-\alpha-2mk,k)\end{array};(-1)^{-k}\ k\ t\ y\right].$ □

5 Differentiation formulas

In this section we give formulas for the differentiation of the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$.

Theorem 5.1. *The following derivative formulas for $\mathbf{H}_{9,k}$ hold true:*

$$D_x^r[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,]=\frac{(\alpha)_{2r,k}}{(\gamma)_{r,k}}\ \mathbf{H}_{9,k}(\alpha+2rk,\beta;\gamma+rk;x,y), \tag{5.1}$$

$$D_y^r[\mathbf{H}_{9,k}(\alpha,\beta,\gamma;\delta;x,y)\,] = \frac{(-1)^{-rk}(\beta)_{r,k}}{(k-\alpha)_{r,k}}\,\mathbf{H}_{9,k}(\alpha-rk,\beta+rk;\gamma;x,y), \tag{5.2}$$

and

$$D_{x,y}^{(r,s)}[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \frac{(\alpha)_{2r-s,k}(\beta)_{s,k}}{(\gamma)_{r,k}}\,\mathbf{H}_{9,k}(\alpha+2rk-sk,\beta+sk;\gamma+rk;x,y). \tag{5.3}$$

Proof. Differentiating (2.1) with respect to x yields

$$D_x[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}(m-1)!n!}x^{m-1}y^n$$
$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+2-n,k}(\beta)_{n,k}}{(\gamma)_{m+1,k}m!n!}x^m y^n = \frac{(\alpha)_{2,k}}{(\gamma)_{1,k}}\,\mathbf{H}_{9,k}(\alpha+2k,\beta;\gamma+k;x,y).$$

Repeating this process, we obtain

$$D_x^r[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \frac{(\alpha)_{2r,k}}{(\gamma)_{r,k}}\,\mathbf{H}_{9,k}(\alpha+2rk,\beta;\gamma+rk;x,y),$$

which is the desired result (5.1).
Differentiating (2.1) with respect to y yields

$$D_y[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}m!(n-1)!}x^m y^{n-1}$$
$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n-1,k}(\beta)_{n+1,k}}{(\gamma)_{m,k}(m)!n!}x^m y^n = (\alpha)_{-1,k}(\beta)_{1,k}\,\mathbf{H}_{9,k}(\alpha-k,\beta+k;\gamma;x,y).$$

Repeating this process, we obtain

$$D_y^r[\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = (\alpha)_{-r,k}(\beta)_{r,k}\,\mathbf{H}_{9,k}(\alpha-rk,\beta+rk;\gamma;x,y)$$
$$= \frac{(-1)^{-rk}(\beta)_{r,k}}{(k-\alpha)_{r,k}}\,\mathbf{H}_{9,k}(\alpha-rk,\beta+rk;\gamma;x,y)$$

which is the desired result (5.2).
Also differentiating (2.1) with respect to x and y, then repeating differentiation with respect to x and y of orders r, s respectively and making some simple calculation yields (5.3).

□

Theorem 5.2. *The following derivative formulas for* (2.1) *hold true:*

$$D_y^r[y^{\frac{\beta}{k}+r-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \frac{(\beta+nk)_{r,k}\;\;y^{\frac{\beta}{k}-1}}{k^r}\,\mathbf{H}_{9,k}(\alpha,\beta+rk,\gamma;\delta;x,y), \tag{5.4}$$

and

$$D_x^r[x^{\frac{\gamma}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \frac{(-1)^{rk}\;(k-\gamma)_{r,k}\;\;x^{\frac{\gamma}{k}-r-1}}{k^r}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma-rk;x,y). \tag{5.5}$$

Proof. Multiplying (2.1) by $y^{\frac{\beta}{k}+r-1}$ and taking the derivative of order r with respect to y, we have

$$D_y^r[y^{\frac{\beta}{k}+r-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = D_y^r[\;y^{\frac{\beta}{k}+r-1}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}m!n!}x^m y^n\;]$$
$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}(\frac{\beta}{k}+n)_r}{(\gamma)_{m,k}m!n!}x^m y^{\frac{\beta}{k}+n-1},$$

Since we have

$$(\frac{\beta}{k}+n)_r = \frac{(\beta+nk)_{r,k}}{k^r} \tag{5.6}$$

Then we obtain

$$D_y^r[y^{\frac{\beta}{k}+r-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}(\beta+nk)_{r,k}}{k^r\ (\gamma)_{m,k} m!n!} x^m y^{\frac{\beta}{k}+n-1}$$

$$= (\beta)_{r,k}\ y^{\frac{\beta}{k}-1} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta+rk)_{n,k}}{k^r\ (\gamma)_{m,k}\ m!n!} x^m y^n = \frac{(\beta+nk)_{r,k}\ \ y^{\frac{\beta}{k}-1}}{k^r}\ \mathbf{H}_{9,k}(\alpha,\beta+rk,\gamma;\delta;x,y).$$

Thus, we obtain (5.4).
Multiplying (2.1) by $x^{\frac{\gamma}{k}-1}$ and taking the derivative of order r with to x, we can write

$$D_x^r[x^{\frac{\gamma}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = D_x^r[\ x^{\frac{\gamma}{k}-1} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k} m!n!}\ x^m y^n\ ]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ (\frac{\gamma}{k}+m)_{-r,k} m!n!}\ x^{\frac{\gamma}{k}+m-r-1} y^n,$$

Since we have,

$$(\frac{\gamma}{k}+m)_{-r} = \frac{(\gamma+mk)_{-r,k}}{k^{-r}} \tag{5.7}$$

Then we obtain

$$D_x^r[x^{\frac{\gamma}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = \sum_{m,n=0}^{\infty} \frac{k^{-r}(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ (\gamma+mk)_{-r,k} m!n!}\ x^{\frac{\gamma}{k}+m-r-1} y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{k^{-r}(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{-r,k}\ (\gamma-rk)_{m,k} m!n!}\ x^{\frac{\gamma}{k}+m-r-1} y^n = \frac{(-1)^{rk}\ (k-\gamma)_{r,k}\ \ x^{\frac{\gamma}{k}-r-1}}{k^r}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma-rk;x,y)$$

which gives (5.5). □

6 Infinite sums

In this section we give some infinite sums formulas for the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$.

Theorem 6.1. *For $|t|<1$, the infinite summation formulas for the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$ hold true:*

$$\sum_{r=0}^{\infty} \frac{(\alpha)_{r,k}}{r!}\ t^r\ \mathbf{H}_{9,k}(\alpha+rk,\beta;\gamma;x,y) = (1-kt)^{-\frac{\alpha}{k}}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\frac{x}{(1-kt)^2}, y(1-kt)) \tag{6.1}$$

and

$$\sum_{r=0}^{\infty} \frac{(\beta)_{r,k}}{r!}\ t^r\ \mathbf{H}_{9,k}(\alpha,\beta+rk;\gamma;x,y) = (1-kt)^{-\frac{\beta}{k}}\ \mathbf{H}_{9,k}(\ \alpha,\beta;\gamma;x,\frac{y}{(1-kt)}\ ). \tag{6.2}$$

Proof. Using the fact that

$$(1-kt)^{-\frac{\alpha}{k}} = \sum_{r=0}^{\infty} \frac{(\alpha)_{r,k}}{k!}\ t^r,$$

We can write

$$(1-kt)^{-\frac{\alpha}{k}}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\frac{x}{(1-kt)^2}, y(1-kt)) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ m!n!}\ x^m y^n (1-kt)^{-\frac{\alpha}{k}+n-2m}$$

$$= \sum_{r,m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\alpha+(2m-n)k)_{r,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ r!m!n!}\ x^m y^n t^k = \sum_{r=0}^{\infty} \frac{(\alpha)_{r,k}}{r!}\ t^r \sum_{m,n=0}^{\infty} \frac{(\alpha+rk)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ m!n!}\ x^m y^n$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_{r,k}}{r!}\ t^r\ \mathbf{H}_{9,k}(\alpha+rk,\beta;\gamma;x,y),$$

which proves the desired result (6.1).
Also we can write

$$(1-kt)^{-\frac{\beta}{k}}\ \mathbf{H}_{9,k}(\ \alpha,\beta;\gamma;x,\frac{y}{(1-kt)}\ )=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}}{(\gamma)_{m,k}\ m!n!}\ x^m y^n(1-kt)^{-\frac{\beta}{k}-n}$$

$$=\sum_{r,m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}(\beta+nk)_{r,k}}{(\gamma)_{m,k}\ r!m!n!}\ x^m y^n t^r=\sum_{r=0}^{\infty}\frac{(\beta)_{r,k}}{r!}\ t^r\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta+rk)_{n,k}}{(\gamma)_{m,k}\ m!n!}\ x^m y^n$$

$$=\sum_{r=0}^{\infty}\frac{(\beta)_{r,k}}{r!}\ t^r\ \mathbf{H}_{9,k}(\alpha,\beta+rk;\gamma;x,y).$$

which proves (6.2). □

7 Recursion formulas for the k- Horn's function $\mathbf{H}_{9,k}$

In this section we present some recursion relations for the k- Horn's function $\mathbf{H}_{9,k}$, let us start with the following theorem.

Theorem 7.1. *For $n\in\mathbb{N}$ and $\gamma\neq 0,-1,-2,....,$ the following recursion formulas for the k- Horn's function $\mathbf{H}_{9,k}$ hold:*

$$\mathbf{H}_{9,k}(\alpha+n,\beta;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)+\frac{2kx}{\gamma}\sum_{r=1}^{n}(\alpha+r)\mathbf{H}_{9,k}(\alpha+r+1,\beta;\gamma+1;x,y)$$
$$-\beta\ ky\sum_{r=1}^{n}\frac{\mathbf{H}_{9,k}(\alpha+r-2,\beta+1;\gamma;x,y)}{(\alpha+r-1)(\alpha+r-2)},\alpha\neq 1-r,\alpha\neq 2-r,r\in\mathbb{N}, \tag{7.1}$$

and

$$\mathbf{H}_{9,k}(\alpha-n,\beta;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)-\frac{2kx}{\gamma}\sum_{r=1}^{n}(\alpha-r-1)\mathbf{H}_{9,k}(\alpha-r+2,\beta;\gamma+1;x,y)$$
$$+\beta\ ky\sum_{r=1}^{n}\frac{\mathbf{H}_{9,k}(\alpha-r-1,\beta+1;\gamma;x,y)}{(\alpha-r)(\alpha-r-1)},\alpha\neq r,\alpha\neq r+1,r\in\mathbb{N}. \tag{7.2}$$

Proof. Applying the transformation formula $(\alpha+1)_{2m-n,k}=(\alpha)_{2m-n,k}(\ 1+\frac{(2m-n)\ k}{\alpha}\ )$in the definition of the k- Horn's function $\mathbf{H}_{9,k}$, we get the contiguous formula:

$$\mathbf{H}_{9,k}(\alpha+1,\beta;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)+\frac{2k(\alpha+1)x}{\gamma}\ \mathbf{H}_{9,k}(\alpha+2,\beta;\gamma+1;x,y)$$
$$-\beta\ ky\frac{\mathbf{H}_{9,k}(\alpha-1,\beta+1;\gamma;x,y)}{\alpha(\alpha-1)},\gamma\neq 0,\alpha\neq 0,\alpha\neq 1. \tag{7.3}$$

Calculating the function $\mathbf{H}_{9,k}$ with the parameter $\alpha+n$ by (7.3) for n times, we obtain the required result (7.1). Replacing α by $\alpha-1$ in (7.3), we get

$$\mathbf{H}_{9,k}(\alpha-1,\beta;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)-\frac{2\ k\ \alpha\ x}{\gamma}\ \mathbf{H}_{9,k}(\alpha+1,\beta;\gamma+1;x,y)$$
$$+\beta\ ky\frac{\mathbf{H}_{9,k}(\alpha-2,\beta+1;\gamma;x,y)}{(\alpha-1)(\alpha-2)},\gamma\neq 0,\alpha\neq 1,\alpha\neq 2. \tag{7.4}$$

Calculating the function $\mathbf{H}_{9,k}$ with the parameter $\alpha-n$ by (7.4) for n times, obtain the required result (7.2). □

Theorem 7.2. *For $\alpha\neq 1$ the following recursion relations hold good for the k- Horn's function $\mathbf{H}_{9,k}$:*

$$\mathbf{H}_{9,k}(\alpha,\beta+n;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)+\frac{k\ y}{\alpha-1}\sum_{r=1}^{n}\mathbf{H}_{9,k}(\alpha-1,\beta+r;\gamma;x,y) \tag{7.5}$$

and

$$\mathbf{H}_{9,k}(\alpha,\beta-n;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)-\frac{k\,y}{\alpha-1}\sum_{r=1}^{n}\mathbf{H}_{9,k}(\alpha-1,\beta-r+1;\gamma;x,y). \tag{7.6}$$

Proof. Applying the transformation formula $(\beta+1)_{n,k}=(\beta)_{n,k}(1+\frac{n\,k}{\beta})$ in the definition of the extension of the k- Horn's function $\mathbf{H}_{9,k}$, we get the contiguous formula:

$$\mathbf{H}_{9,k}(\alpha,\beta+1;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)+\frac{k\,y}{\alpha-1}\,\mathbf{H}_{9,k}(\alpha-1,\beta+1;\gamma;x,y). \tag{7.7}$$

Calculating the function $\mathbf{H}_{9,k}$ with the parameter $\beta+n$ by (7.7) for n times, we obtain the required result (7.5). Replacing β by $\beta-1$ in (7.7), we get

$$\mathbf{H}_{9,k}(\alpha,\beta-1;\gamma;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)-\frac{k\,y}{\alpha-1}\,\mathbf{H}_{9,k}(\alpha-1,\beta;\gamma;x,y). \tag{7.8}$$

Calculating the function $\mathbf{H}_{9,k}$ with the parameter $\beta-n$ by (7.8) for (7.6) n times, we obtain the required result. □

Theorem 7.3. *The k- Horn's hypergeometric function $\mathbf{H}_{9,k}$ satisfies the identity:*

$$\begin{aligned}&\mathbf{H}_{9,k}(\alpha,\beta;\gamma-n;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\\&+\alpha(\alpha+1)\,k\,x\sum_{r=1}^{n}\frac{\mathbf{H}_{9,k}(\alpha+2,\beta;\gamma-r+2;x,y)}{(\gamma-r)(\gamma-r+1)},\ \gamma\neq r-1,\gamma\neq r,r\in\mathbb{N},\end{aligned} \tag{7.9}$$

and

$$\begin{aligned}&\mathbf{H}_{9,k}(\alpha,\beta;\gamma+n;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\\&-\alpha(\alpha+1)\,k\,x\sum_{r=1}^{n}\frac{\mathbf{H}_{9,k}(\alpha+2,\beta;\gamma+r+1;x,y)}{(\gamma+r)(\gamma+r-1)},\ \gamma\neq -r,\gamma\neq 1-r,r\in\mathbb{N}.\end{aligned} \tag{7.10}$$

Proof. From the relation

$$\frac{1}{(\gamma-1)_{m,k}}=\frac{1}{(\gamma)_{m,k}}+\frac{m\,k}{(\gamma-1)(\gamma)_{m,k}},$$

we can write

$$\begin{aligned}&\mathbf{H}_{9,k}(\alpha,\beta;\gamma-1;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\\&+\frac{\alpha(\alpha+1)\,k\,x}{(\gamma-1)\gamma}\ \mathbf{H}_{9,k}(\alpha+2,\beta;\gamma+1;x,y),\gamma\neq 0,1.\end{aligned} \tag{7.11}$$

Iterating this method on $\mathbf{H}_{9,k}$ with the parameter $\gamma-n$ for n times, we obtain (7.9). Replacing γ by $\gamma+1$ in (7.11), we obtain

$$\begin{aligned}&\mathbf{H}_{9,k}(\alpha,\beta;\gamma+1;x,y)=\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\\&-\frac{\alpha(\alpha+1)\,k\,x}{\gamma(\gamma+1)}\ \mathbf{H}_{9,k}(\alpha+2,\beta;\gamma+2;x,y),\gamma\neq 0,-1.\end{aligned} \tag{7.12}$$

Similarly iterating this method on $\mathbf{H}_{9,k}$ with the parameter $\gamma+n$ for n times, we obtain (7.10). □

8 Laplace, Mellin and Fractional Fourier Transforms

In this section we obtain the Laplace, Mellin and Fractional Fourier transforms of $\mathbf{H}_{9,k}$. Before embarking upon that mission we find it very pertinent to remark here that the second author has recently given the most versatile generalization of the celebrated Laplace transform and its numerous other variants that are available in the mathematics research literature by the name Upadhyaya transform (see, Upadhyaya [17] and Upadhyaya et al. [18]). Thus the expressions for the Laplace transform of the function $\mathbf{H}_{9,k}$ that we give below can also more efficiently be dealt with by taking the Upadhyaya transform of the function $\mathbf{H}_{9,k}$, which can be a further interesting direction of research in this field.
Now we give below three theorems, one for each about the Laplace, Mellin and Fractional Fourier transforms of $\mathbf{H}_{9,k}$.

Theorem 8.1. *The Laplace transform of* $\mathbf{H}_{9,k}$ *is as follows :*

$$\mathcal{L}\{x^{\frac{a}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\}=\frac{k\,\Gamma_k(a)}{(ks)^{\frac{a}{k}}}\,{}_3F_{1,k}\left[\begin{matrix}\left(\frac{\alpha}{2},k\right),\left(\frac{\alpha+k}{2},k\right),(a,k)\\(\gamma,k)\end{matrix};\frac{4\omega}{ks}\right]\times \\ {}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(k-\alpha-2mk,k)\end{matrix};(-1)^{-k}\,y\right]. \tag{8.1}$$

Proof. Taking the Laplace transform (see, (1.4)) of the function $x^{\frac{a}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y)$ with respect to the parameter s, we have

$$\mathcal{L}\{x^{\frac{a}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\}=\int_0^\infty x^{\frac{a}{k}-1}e^{-sx}[\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}(\omega x)^m y^n}{(\gamma)_{m,k}m!n!}]dx.$$

Putting $sx=u$, $dx=\frac{du}{s}$, $x=0$, $u=0$ and $x=\infty$, $u=\infty$, we get

$$\mathcal{L}\{x^{\frac{a}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\}=\sum_{m,n=0}^{\infty}\left[\frac{1}{s^{\frac{a}{k}+m}}\left(\int_0^\infty e^{-u}u^{\frac{a}{k}+m-1}du\right)\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}\omega^m y^n}{(\gamma)_{m,k}m!n!}\right]$$
$$=\frac{1}{s^{\frac{a}{k}}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}\Gamma\left(\frac{a}{k}+m\right)\omega^m y^n}{s^m(\gamma)_{m,k}m!n!},$$

Since we have

$$(\alpha)_{2m-n,k}=(\alpha)_{2m,k}(\alpha+2mk)_{-n,k}=\frac{(-1)^{-nk}\,2^{2m}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}}{(k-\alpha-2mk)_{n,k}},$$

and

$$\Gamma(\frac{a}{k}+m)=k^{1-\frac{a+mk}{k}}\;\Gamma_k(a+mk),$$

Then, we can write

$$\mathcal{L}\{x^{\frac{a}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\}=\frac{k\,\Gamma_k(a)}{(ks)^{\frac{a}{k}}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}(\beta)_{n,k}\,(a)_{m,k}\,\omega^m y^n}{(ks)^m\,(\gamma)_{m,k}m!n!}$$
$$=\frac{k\,\Gamma_k(a)}{(ks)^{\frac{a}{k}}}\sum_{m,n=0}^{\infty}\frac{4^m\;(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\,(a)_{m,k}\,\omega^m y^n}{(ks)^m\,(\gamma)_{m,k}\,(k-\alpha-2mk)_{n,k}\,m!n!}$$
$$=\frac{k\,\Gamma_k(a)}{(ks)^{\frac{a}{k}}}\,{}_3F_{1,k}\left[\begin{matrix}\left(\frac{\alpha}{2},k\right),\left(\frac{\alpha+k}{2},k\right),(a,k)\\(\gamma,k)\end{matrix};\frac{4\omega}{ks}\right]\times\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(k-\alpha-2mk,k)\end{matrix};(-1)^{-k}\,y\right].$$

which is the desired result of (8.1). □

Theorem 8.2. *The Mellin transform of* $\mathbf{H}_{9,k}$ *is as follows :*

$$M\{e^{-x}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\} \\ =\frac{k\,\Gamma_k(ks)}{k^s}\,{}_3F_{1,k}\left[\begin{matrix}\left(\frac{\alpha}{2},k\right),\left(\frac{\alpha+k}{2},k\right),(ks,k)\\(\gamma,k)\end{matrix};\frac{4\omega}{k}\right]\times\,{}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(k-\alpha-2mk,k)\end{matrix};(-1)^{-k}\,y\right]. \tag{8.2}$$

Proof. Taking the Mellin transform (see, (1.5)) of the function $e^{-x}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y)$ with respect to the parameter s, we have

$$M\{e^{-x}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\}=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,\omega^m y^n}{(\gamma)_{m,k}m!n!}\int_0^\infty x^{s-1}e^{-x}x^m dx$$
$$=\sum_{m,n=0}^{\infty}\frac{4^m\;(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\,\Gamma(s+m)\,\omega^m y^n}{(\gamma)_{m,k}\,(k-\alpha-2mk)_{n,k}\,m!n!},$$

From the relation

$$\Gamma(s+m) = k^{1-s-m}\ \Gamma_k(ks+mk),$$

we can write

$$M\{e^{-x}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,y):s\} = \frac{k\ \Gamma_k(ks)}{k^s}\ \sum_{m,n=0}^{\infty} \frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\ (ks)_{m,k}\ \omega^m y^n}{k^m\ (\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}$$

$$= \frac{k\ \Gamma_k(ks)}{k^s}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\frac{\alpha+k}{2},k)\,,(ks,k)\\ (\gamma,k)\end{array};\frac{4\omega}{k}\right]\times\ {}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right].$$

which proves (8.2). □

Theorem 8.3. *The Fractional Fourier transform of $\mathbf{H}_{9,k}$ of order v for $x<0$ is given by:*

$$\Im_v[\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = (i\omega^{\frac{1}{v}})^{-1}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\frac{\alpha+k}{2},k)\,,(k,k)\\ (\gamma,k)\end{array};\frac{-4\ (i\omega^{\frac{1}{v}})^{-1}}{k}\right]\times$$
$${}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right]. \tag{8.3}$$

Proof. Since we have

$$\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ x^m y^n}{(\gamma)_{m,k}m!n!} = \sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\ x^m y^n}{(\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}, \tag{8.4}$$

taking the Fractional Fourier transform (see, (1.6)) of (8.4), we have

$$\Im_v[\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = \sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\ y^n}{(\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}\ \int_{-\infty}^{0} e^{i\omega^{\frac{1}{v}}x}\ x^m dx.$$

Putting $t=-i\omega^{\frac{1}{v}}x$, $dt=-i\omega^{\frac{1}{v}}dx$, $x=-\infty$, $t=\infty$ and $x=0$, $t=0$, we get

$$\Im_v[\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = \sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\ y^n}{(\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}\ \int_{0}^{\infty} e^{-t}\ (-\frac{t}{i\omega^{\frac{1}{v}}})^m\ \frac{dt}{i\omega^{\frac{1}{v}}}$$

$$= \sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\ y^n(-1)^m(i)^{-(m+1)}(\omega)^{-\frac{m+1}{v}}\Gamma(m+1)}{(\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}.$$

From the relation

$$\Gamma(m+1) = k^{1-m-1}\ \Gamma_k(k+mk),$$

we can write

$$\Im_v[\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = \sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}(k)_{m,k}(\beta)_{n,k}\ y^n(-1)^m(i)^{-(m+1)}(\omega)^{-\frac{m+1}{v}}}{k^m\ (\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}$$

$$= (i\omega^{\frac{1}{v}})^{-1}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\frac{\alpha+k}{2},k)\,,(k,k)\\ (\gamma,k)\end{array};\frac{-4\ (i\omega^{\frac{1}{v}})^{-1}}{k}\right]\times\ {}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right].$$

which establishes (8.3).

Special case: For $v=1$ the result shows the conventional Fourier transform

$$\Im[\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\ ] = (i\omega)^{-1}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\frac{\alpha+k}{2},k)\,,(k,k)\\ (\gamma,k)\end{array};\frac{-4\ (i\omega)^{-1}}{k}\right]$$
$$\times\ {}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\ y\right].$$

□

9 Double Laplace and double Mellin transforms

In this section we investigate the double Laplace and double Mellin transforms of $\mathbf{H}_{9,k}$. We mention that the double Laplace transform is a special case of the more general double Upadhyaya transform introduced by the second author in 2019 (see, Upadhyaya [17] and Upadhyaya et al. [18]), therefore, *the results of this section can also be further explored from the view point of the Double Upadhyaya transform, which is an interesting open problem for further research in this direction.* For the results pertaining to the double Laplace and double Mellin transforms of the function $\mathbf{H}_{9,k}$ we prove the following two theorems:

Theorem 9.1. *The double Laplace transform of the k- Horn's function* $\mathbf{H}_{9,k}$ *is as follows :*

$$\mathcal{L}_2\{\ x^{\frac{a}{k}-1}y^{\frac{b}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y)\ \} = \frac{k^2\ \Gamma_k(a)\Gamma_k(b)}{(kr)^{\frac{a}{k}}(ks)^{\frac{b}{k}}}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k),(a,k)\\(\gamma,k)\end{array};\frac{4\omega}{kr}\right]\times$$
$${}_2F_{1,k}\left[\begin{array}{c}(\beta,k),(b,k)\\(k-\alpha-2mk,k)\end{array};\frac{(-1)^{-k}\rho}{ks}\right]. \tag{9.1}$$

Proof. Taking the double Laplace transform (see, (1.7)) of the function $x^{\frac{a}{k}-1}y^{\frac{b}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y)$ we have,

$$\mathcal{L}_2\{\ x^{\frac{a}{k}-1}y^{\frac{b}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y)\ \}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ \omega^m\rho^n}{(\gamma)_{m,k}m!n!}\{\ (\ \int_0^{\infty}x^{\frac{a}{k}+m-1}e^{-rx}dx\ )(\ \int_0^{\infty}y^{\frac{b}{k}+n-1}e^{-sy}dy\ )\ \}.$$

Putting $rx=u$, $dx=\frac{du}{r}$, $x=0$, $u=0$, $x=\infty$, $u=\infty$ and $sy=z$,$dy=\frac{dz}{s}$,$y=0$ $z=0$, $y=\infty$, $z=\infty$ in the above relation we get,

$$\mathcal{L}_2\{\ x^{\frac{a}{k}-1}y^{\frac{b}{k}-1}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y)\ \}$$
$$=\frac{1}{r^{\frac{a}{k}}s^{\frac{b}{k}}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ \omega^m\rho^n}{r^m\ s^n\ (\gamma)_{m,k}m!n!}\{\ (\ \int_0^{\infty}u^{\frac{a}{k}+m-1}e^{-u}du\ )(\ \int_0^{\infty}z^{\frac{b}{k}+n-1}e^{-z}dz\ )\ \}$$
$$=\frac{1}{r^{\frac{a}{k}}s^{\frac{b}{k}}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\ (\beta)_{n,k}\ \Gamma(\frac{a}{k}+m)\Gamma(\frac{b}{k}+n)\omega^m\rho^n}{r^m\ s^n\ (\gamma)_{m,k}m!n!}$$
$$=\frac{k^2\ \Gamma_k(a)\Gamma_k(b)}{(kr)^{\frac{a}{k}}(ks)^{\frac{b}{k}}}\sum_{m,n=0}^{\infty}\frac{4^m\ \ (-1)^{-nk}\ (\frac{\alpha}{2})_{m,k}\ (\frac{\alpha+k}{2})_{m,k}\ (\beta)_{n,k}\ (a)_{m,k}(b)_{n,k}\ \omega^m\rho^n}{(kr)^m(ks)^n\ (\gamma)_{m,k}\ (k-\alpha-2mk)_{n,k}\ m!n!}$$
$$=\frac{k^2\ \Gamma_k(a)\Gamma_k(b)}{(kr)^{\frac{a}{k}}(ks)^{\frac{b}{k}}}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k),(a,k)\\(\gamma,k)\end{array};\frac{4\omega}{kr}\right]\times$$
$${}_2F_{1,k}\left[\begin{array}{c}(\beta,k),(b,k)\\(k-\alpha-2mk,k)\end{array};\frac{(-1)^{-k}\rho}{ks}\right]$$

which establishes the result of (9.1). □

Theorem 9.2. *The double Mellin transform of* $\mathbf{H}_{9,k}$ *is as follows :*

$$M_{xy}\{e^{-(x+y)}\ \mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y):r,s\}$$
$$=\frac{k^2\ \Gamma_k(kr)\Gamma_k(ks)}{k^{r+s}}\ {}_3F_{1,k}\left[\begin{array}{c}(\frac{\alpha}{2},k),(\frac{\alpha+k}{2},k),(kr,k)\\(\gamma,k)\end{array};\frac{4\omega}{k}\right]\times$$
$${}_2F_{1,k}\left[\begin{array}{c}(\beta,k),(ks,k)\\(k-\alpha-2mk,k)\end{array};\frac{(-1)^{-k}\rho}{k}\right] \tag{9.2}$$

Proof. Taking the double Mellin transform (see, (1.8)) of the function $e^{-(x+y)}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y)$ w.r.t. the parameters r,s, we have

$$
\begin{aligned}
&M_{xy}\{e^{-(x+y)}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;\omega x,\rho y):r,s\}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,\omega^m\rho^n}{(\gamma)_{m,k}m!n!}\{\,(\int_0^\infty x^{r-1}e^{-x}x^m dx\,)(\int_0^\infty y^{s-1}e^{-y}y^n dy\,)\,\}\\
&=\sum_{m,n=0}^{\infty}\frac{4^m(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}(\beta)_{n,k}\,\Gamma(r+m)\,\Gamma(s+n)\,\omega^m\rho^n}{(\gamma)_{m,k}\,(k-\alpha-2mk)_{n,k}\,m!n!}\\
&=\frac{k^2\,\Gamma_k(kr)\Gamma_k(ks)}{k^{r+s}}\sum_{m,n=0}^{\infty}\frac{4^m\,(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}(kr)_{m,k}(\beta)_{n,k}\,(b)_{n,k}(ks)_{n,k}\,\omega^m\rho^n}{k^{m+n}\,(\gamma)_{m,k}\,(k-\alpha-2mk)_{n,k}\,m!n!}\\
&=\frac{k^2\,\Gamma_k(kr)\Gamma_k(ks)}{k^{r+s}}\,{}_3F_{1,k}\left[\begin{matrix}(\frac{\alpha}{2},k)\,,(\,\frac{\alpha+k}{2},k)\,,(kr,k)\\(\gamma,k)\end{matrix}\;;\frac{4\omega}{k}\right]\times\\
&{}_2F_{1,k}\left[\begin{matrix}(\beta,k)\,,(ks,k)\\(k-\alpha-2mk,k)\end{matrix}\;;\frac{(-1)^{-k}\,\rho}{k}\right].
\end{aligned}
$$

which proves (9.2). □

10 Fractional integration and k-Fractional differentiation

We investigate the fractional integration and k-fractional differentiation formulas of $\mathbf{H}_{9,k}$ in this section.

Theorem 10.1. *Let $\xi,\varrho,\eta\in\mathbb{C}$ with $\Re(\xi)>0$. Then the fractional integration of the k- Horn's function $\mathbf{H}_{9,k}$ is as follows:*

$$
\begin{aligned}
&I_{0,x}^{\xi,\varrho,\eta}[x^{\frac{\rho}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,]\\
&=k^{\xi}\,x^{\frac{\rho}{k}-\varrho-1}\,\frac{\Gamma_k(\rho)\,\Gamma_k(\rho+\eta k-\varrho k)}{\Gamma_k(\rho-\varrho k)\,\Gamma_k(\rho+\eta k+\xi k)}\,{}_4F_{3,k}\left[\begin{matrix}(\frac{\alpha}{2},k)\,,(\,\frac{\alpha+k}{2},k)\,,(\rho,k)\;,(\rho+\eta k-\varrho k,k)\\(\gamma,k)\,,(\rho-\varrho k,k)\;,(\rho+\eta k+\xi k,k)\end{matrix}\;;4x\right]\times\\
&{}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(k-\alpha-2mk,k)\end{matrix}\;;(-1)^{-k}\,y\right]
\end{aligned}
\tag{10.1}
$$

Proof. From Saigo and Maeda [5] we have,

$$
I_{0,x}^{\xi,\varrho,\eta}x^{\lambda-1}=x^{\lambda-\varrho-1}\,\frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\varrho)}{\Gamma(\lambda-\varrho)\Gamma(\lambda+\eta+\xi)}.
$$

Then we can write

$$I_{0,x}^{\xi,\varrho,\eta}[x^{\frac{\rho}{k}-1}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,y^n}{(\gamma)_{m,k}\,m!n!}\;\; I_{0,x}^{\xi,\varrho,\eta}[x^{\frac{\rho}{k}+m-1}]$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}\,y^n}{(\gamma)_{m,k}\,m!n!}\;[\,x^{\frac{\rho}{k}-\varrho+m-1}\,\frac{\Gamma(\frac{\rho}{k}+m)\Gamma(\frac{\rho}{k}+\eta-\varrho+m)}{\Gamma(\frac{\rho}{k}-\varrho+m)\Gamma(\frac{\rho}{k}+\eta+\xi+m)}\,]$$
$$= x^{\frac{\rho}{k}-\varrho-1}\sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}(\beta)_{n,k}\,x^m\,y^n}{(\gamma)_{m,k}\,m!n!} \times \frac{k^{1-\frac{\rho+mk}{k}}\,\Gamma_k(\rho+mk)}{k^{1-\frac{\rho-\varrho k+mk}{k}}\,\Gamma_k(\rho-\varrho k+mk)}$$
$$\times \frac{k^{1-\frac{\rho+\eta k-\varrho k+mk}{k}}\,\Gamma_k(\rho+\eta k-\varrho k+mk)}{k^{1-\frac{\rho+\eta k+\xi k+mk}{k}}\,\Gamma_k(\rho+\eta k+\xi k+mk)}$$
$$= k^{\xi}\,x^{\frac{\rho}{k}-\varrho-1}\,\frac{\Gamma_k(\rho)\,\Gamma_k(\rho+\eta k-\varrho k)}{\Gamma_k(\rho-\varrho k)\,\Gamma_k(\rho+\eta k+\xi k)}\times$$
$$\sum_{m,n=0}^{\infty} \frac{4^m\,(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}\,(\rho)_{m,k}\,(\rho+\eta k-\varrho k)_{m,k}\,(\beta)_{n,k}\,x^m\,y^n}{(\gamma)_{m,k}\,(\rho-\varrho k)_{m,k}\,(\rho+\eta k+\xi k)_{m,k}\,(k-\alpha-2mk)_{n,k}\,m!n!}$$
$$= k^{\xi}\,x^{\frac{\rho}{k}-\varrho-1}\,\frac{\Gamma_k(\rho)\,\Gamma_k(\rho+\eta k-\varrho k)}{\Gamma_k(\rho-\varrho k)\,\Gamma_k(\rho+\eta k+\xi k)}$$
$$\times\,{}_4F_{3,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\,\frac{\alpha+k}{2},k)\,,(\rho,k)\;,(\rho+\eta k-\varrho k,k)\\ (\gamma,k)\,,(\rho-\varrho k,k)\;,(\rho+\eta k+\xi k,k)\end{array};4x\right]\times$$
$${}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\,y\right].$$

which gives the desired result of (10.1). □

Theorem 10.2. *Let μ be a real number and $0 \prec \mu \preceq 1$. Then the k-Fractional differentiation of the k- Horn's function $\mathbf{H}_{9,k}$ is as follows:*

$$D_k^{\mu}[x^{\frac{\lambda}{k}}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,]$$
$$= \frac{\lambda\,x^{\frac{1-\mu+\lambda}{k}-1}\,\Gamma_k(\,\lambda\,)}{k\,\Gamma_k(\,1-\mu+\lambda)}\,{}_3F_{2,k}\left[\begin{array}{c}(\frac{\alpha}{2},k)\,,(\,\frac{\alpha+k}{2},k)\,,(\lambda+k,k)\\ (\gamma,k)\,,(1-\mu+\lambda,k)\end{array};4x\right] \tag{10.2}$$
$$\times\,{}_1F_{1,k}\left[\begin{array}{c}(\beta,k)\\ (k-\alpha-2mk,k)\end{array};(-1)^{-k}\,y\right].$$

Proof. Applying the k-Fractional differentiation (see, (1.10)), we have

$$D_k^{\mu}[x^{\frac{\lambda}{k}}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,] = \frac{d}{dx}\,[\,I_k^{1-\mu}\,x^{\frac{\lambda}{k}}\,\mathbf{H}_{9,k}(\alpha,\beta;\gamma;x,y)\,]$$
$$= \frac{d}{dx}\,\frac{1}{k\,\Gamma_k(1-\mu)}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,y^n}{(\gamma)_{m,k}\,m!n!}\;[\int_0^x t^{\frac{\lambda}{k}+m}\,(x-t)^{\frac{1-\mu}{k}-1}\,dt\,].$$

Putting $t = xz$, $dt = xdz$, $t = 0$, $z = 0$ and $t = x$, $z = 1$, we get

$$D_k^{\mu}[x^{\frac{\lambda}{k}}\, \mathbf{H}_{9,k}(\alpha, \beta; \gamma; x, y)\,]$$
$$= \frac{d}{dx}\,\frac{1}{k\,\Gamma_k(1-\mu)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\, x^{\frac{1-\mu+\lambda+mk}{k}}\; y^n}{(\gamma)_{m,k}\; m!n!}\;\Big[\int_0^1 z^{\frac{\lambda}{k}+m}\,(1-z)^{\frac{1-\mu}{k}-1}\, dz\,\Big]$$
$$= \frac{1}{k\,\Gamma_k(1-\mu)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,(\,\frac{1-\mu+\lambda+mk}{k}\,)\, x^{\frac{1-\mu+\lambda+mk}{k}-1}\, B(\,\frac{1-\mu}{k}, \frac{\lambda}{k}+m+1\,)}{(\gamma)_{m,k}\; m!n!}$$
$$= \frac{1}{k\,\Gamma_k(1-\mu)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,(\,\frac{1-\mu+\lambda+mk}{k}\,)\, x^{\frac{1-\mu+\lambda+mk}{k}-1}\, y^n\,\Gamma(\,\frac{1-\mu}{k}\,)\,\Gamma(\,\frac{\lambda+mk+k}{k}\,)}{(\gamma)_{m,k}\,\Gamma(\,\frac{1-\mu+\lambda+mk}{k}+1\,)\; m!n!}$$
$$= \frac{1}{k\,\Gamma_k(1-\mu)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\,(\,\frac{1-\mu+\lambda+mk}{k}\,)\; x^{\frac{1-\mu+\lambda+mk}{k}-1}\, y^n\,\Gamma(\,\frac{1-\mu}{k}\,)\,\Gamma(\,\frac{\lambda+mk+k}{k}\,)}{(\gamma)_{m,k}\,(\,\frac{1-\mu+\lambda+mk}{k}\,)\,\Gamma(\,\frac{1-\mu+\lambda+mk}{k}\,)\; m!n!}$$
$$= \frac{1}{k\,\Gamma_k(1-\mu)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n,k}\,(\beta)_{n,k}\; x^{\frac{1-\mu+\lambda+mk}{k}-1}\, y^n\; k^{1-\frac{1-\mu}{k}}\; k^{1-\frac{\lambda+mk+k}{k}}\;\Gamma_k(\,1-\mu\,)\,\Gamma_k(\,\lambda+mk+k\,)}{(\gamma)_{m,k}\; k^{1-\frac{1-\mu+\lambda+mk}{k}}\,\Gamma_k(\,1-\mu+\lambda+mk\,)\; m!n!}$$
$$= \frac{x^{\frac{1-\mu+\lambda}{k}-1}\,\Gamma_k(\,\lambda+k\,)}{k\,\Gamma_k(\,1-\mu+\lambda)} \sum_{m,n=0}^{\infty} \frac{4^m\;\;(-1)^{-nk}\,(\frac{\alpha}{2})_{m,k}\,(\frac{\alpha+k}{2})_{m,k}\,(\lambda+k)_{m,k}\,(\beta)_{n,k}\; x^m\; y^n}{(\gamma)_{m,k}\;(1-\mu+\lambda)_{m,k}\;(k-\alpha-2mk)_{n,k}\; m!n!}$$
$$= \frac{\lambda\, x^{\frac{1-\mu+\lambda}{k}-1}\,\Gamma_k(\,\lambda\,)}{k\,\Gamma_k(\,1-\mu+\lambda)}\;{}_3F_{2,k}\left[\begin{matrix}(\frac{\alpha}{2},k)\,,(\,\frac{\alpha+k}{2},k)\,,(\lambda+k,k)\\ (\gamma,k)\,,(1-\mu+\lambda,k)\end{matrix}\;;4x\right]\times$$
$${}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\ (k-\alpha-2mk,k)\end{matrix}\;;(-1)^{-k}\,y\right].$$

which proves (10.2). □

Special case: For $k = 1$ the result gives the fractional differentiation of the Horn's hypergeometric function H_9

$$D^{\mu}[x^{\lambda}\,H_9(\alpha, \beta; \gamma; x, y)\,]$$
$$= \frac{\lambda\,\Gamma(\,\lambda\,)\,x^{\lambda-\mu}}{\Gamma(\,1-\mu+\lambda)}\;{}_3F_2(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \lambda+1; \gamma\,, 1-\mu+\lambda; 4x)\;{}_1F_1(\beta\,; 1-\alpha-2m\,; -y).$$

11 Concluding remarks

In this paper we explored some interesting properties of the k- Horn's hypergeometric function $\mathbf{H}_{9,k}$ which study can be extended to the other hypergeometric functions of this class.

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