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Two Finite Simple Directed Graphs

A. Rameshkumar[1,*], G. Kavitha[2]

Author Affiliation:
[1] Assistant Professor, Department of Mathematics, Srimad Andavan Arts and Science College (Autonomous) No.7, Nelson Road, T.V. Kovil, Trichy, Tamilnadu, India-620005.
E-mail: andavanmathsramesh@gmail.com
[2]Assistant Professor, Kongunadu College of Engineering and Technology, Tholurpatti, Hottiam.
E-mail: kavipraba18@gmail.com

***Corresponding Author:**
Dr. A. Rameshkumar, Head & Research Adviser, Department of Mathematics, Srimad Andavan Arts and Science College (Autonomous) No.7, Nelson Road, T.V. Kovil, Trichy, Tamilnadu, India-620005.

E-mail: andavanmathsramesh@gmail.com

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Abstract

The main result of the paper is that in practically interesting situations determining the optimal number of colors in the proposed coloring is an NP-hard problem. A possible conclusion to draw from this result is that for practical transitive tournament search algorithms we have to develop approximate greedy coloring algorithms. It is an empirical fact that coloring the nodes of a graph can be used to speed up clique search algorithms. In directed graphs transitive sub tournaments can play the role of cliques. In order to speed up algorithms to locate large transitive tournaments we propose a scheme for coloring the nodes of a directed graph.

Keywords: Directed graph, tournament, Graph coloring, NP-hard problem.

Mathematical Subject Classification: 05C15

1. INTRODUCTION

Let G = (V; E) be a finite simple graph, that is, G has finitely many nodes and G does not have any loop or double edge. A subgraph D is a clique in G if each two distinct nodes of D are connected in G. If the clique D has k nodes, then we say that D is a k-clique in G. The number of nodes of a clique is sometimes referred to as the size of the clique. A k-clique in G is a maximum clique if G does not have any (k + 1)-clique. The graph G may have several maximum cliques but their common size is a well defined number. This number is called the clique number of the graph G and it is denoted by $\omega(G)$. The problem of determining the clique number of a given graph is an important problem in many areas of applied discrete mathematics [1]. It is known that the problem is NP-complete [5]. The most commonly used clique search algorithms employ coloring of the nodes of a graph to speed up the computations [2, 8, 9]. The coloring of the nodes of the graph G with k colors assigns exactly one color to each node of the graph such that adjacent nodes never receive the same color [3]. This type of coloring of the nodes is sometimes referred to as legal or well coloring of the nodes. The minimum number of colors with which the nodes of G can be legally colored is a well defined number. It is called the chromatic number of G and it is denoted by (G). Determining the chromatic number of a given graph is another important problem in the applied discrete mathematics with many applications [6]. It is known that the problem of deciding if a given graph can be colored with k color is NP-complete for any fixed k, where $k \geq 4$.

Colorings are employed in at least two ways in clique search algorithms [7]. One can color the nodes of a graph G before the clique search starts. In these cases the coloring is used as a possible preprocessing or preconditioning tool. On the other hand if in the course of the clique search algorithm [4] one recolors the nodes of the subgraphs of G under consideration, then we call it an on-line coloring [10], for example, the optimization problem of finding the least cost cyclic route through all the nodes of a weighted graph.

2. COLORING THE NODES

Let T be a tournament whose nodes are $a_1; \ldots, a_k$, where $k \geq 2$.

Lemma 1 : If T is a transitive tournament, then there is a permutation $b_1, \ldots, b_k$ of $a_1, \ldots, a_k$ such that

(1) $(b_i, b_{i+1}), \ldots, (b_i, b_k)$ are edges of T for each i, $1 \leq i < k$.

(2) The listed $n(n - 1)/2$ edges are all the edges of T.

(3) Each subgraph of T is a transitive tournament.

Proof : Statement (1) clearly holds for k = 2. We assume that k =3 and start an induction on k. If T has a vertex, say a_1, such that each edge incident to a_1 goes out of a_1, then a_1 can be identified with b_1 and the inductive assumption is applicable to the graph whose nodes are $a_2, \ldots, a_k$.
If T has a vertex, say a_k such that each edge incident to a_k goes into a_k, then a_k can be identified with b_k and the inductive assumption is applicable to the graph whose nodes are $a_1, \ldots, a_{k-1}$.
We may assume that each vertex of T has an incident edge going in and has an incident edge going out. In this case T contains a directed cycle. On the other hand a transitive tournament cannot contain a directed cycle.

The reason why statement (2) holds is that a transitive tournament has $n(n - 1)/2$ directed edges and we list all of them in statement (1).

Statement (3) follows from the definition of the transitive tournament and statement (1), as any subset of the nodes can be ordered the same way.

3. TWO AUXILIARY GRAPHS

We describe two finite simple directed graphs. They will play the roles of building blocks or switching devices in further constructions. Let r, s be fixed integers such that $r \geq 4$, $s \geq 4$. Set

$$h = (s - 1)(r - 1) + (s - 2)(r - 2) + (s - 3) + 3$$

Let us consider the directed simple graph H = (V, E), where V = {1, … , h}. Set W = {2, . . . , h – 1}. We draw directed edges between the nodes in W such that the subgraph of H whose set of nodes is W forms a transitive tournament. From the node 1 we direct edges towards each node of W. Similarly, from the node h we direct edges towards each node of W.

We worked out the special case r = 4, s = 4 in detail. The adjacency matrix of H is shown in Table 1. A geometric representation of H is depicted in Figure 1.

	1	2	3	4	5	6	7	8	9
1		•	•	•	•	•	•	•	
2			•	•	•	•	•	•	
3				•	•	•	•	•	
4					•	•	•	•	
5						•	•	•	
6							•	•	
7								•	
8									
9		•	•	•	•	•	•	•	

Table 1: The adjacency matrix of the graph H in the special case r = s = 4

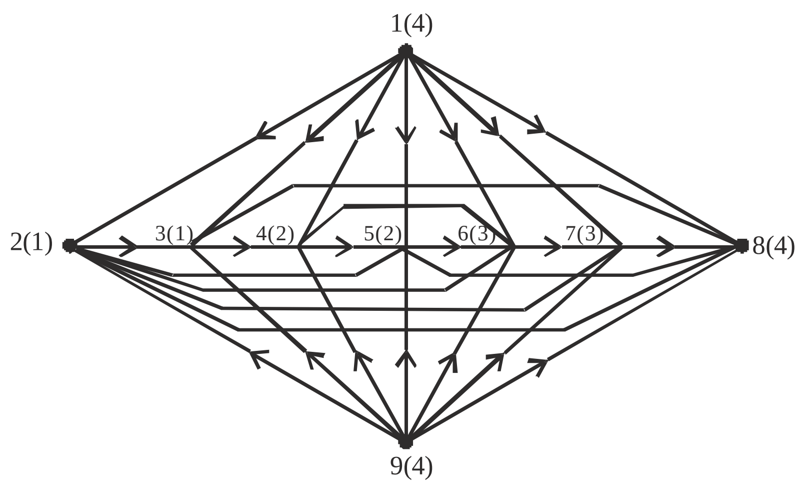

Figure 1: The auxiliary graph H in the special case r = s = 4. The numbers in parentheses are the colors of the nodes.

Lemma 3.1.

(1) The nodes of the graph K admit an s-free coloring with r colors.

(2) In each s-free coloring of the nodes of the graph K with r colors the nodes 1 and k cannot receive the same colors.

(3) Each partial coloring of the nodes of K, where the nodes 1 and k are colored with distinct colors (and the other nodes of K are left uncolored) can be extended to an s-free coloring of the nodes of K using r colors.

Proof : The nodes of the graph H admit an s-free coloring with r colors. Consequently, the nodes of each of the graphs $H_1, \ldots, H_{s-1}$ admit an s-free coloring with r colors. These colorings provide the same fixed color for s - 1 nodes of the tournament T. The uncolored node v of T can be colored with any of the remaining r - 1 colors.

By a Lemma, the node V cannot receive the same color as node u.

Further by another Lemma, each partial coloring of the nodes of H can be extended to an s-free coloring of the nodes of H using r colors. It follows that each partial coloring of the nodes of H_i can be extended to an s-free coloring of the nodes of H_i using r colors for each i, $1 \leq i \leq s-1$. This provides a partial coloring of the nodes of the tournament T. In this partial coloring of the nodes of T each node except node v receives the same color. Namely the color of node u. The last uncolored node v clearly can be colored with any of the remaining r - 1 colors.

4. THE MAIN RESULT

Theorem 4.1

A finite simple directed graph G = (V, E). Further given integers r, s such that $r \geq 4$, $s \geq 3$. Decide if G has an s-free coloring with r colors. When we deal with coloring the nodes of a graph G we inevitably have to deal with incomplete or partial colorings, where each of the nodes of G receives at most one of the colors 1, . . . , r but some of the nodes of G are left uncolored. Allocating color 0 for the uncolored nodes we can incorporate the incomplete colorings into the family of complete colorings.

First Proof : Let r, $s \geq 4$ be fixed integers. Let G = (V; E) be a finite simple graph with undirected edges. Using G and the auxiliary graphs H, K described in the previous section we construct a finite simple directed graph $G_0 = (V_0; E_0)$ such that the following conditions hold.

(1) If the nodes of G′ have an s-free coloring with r colors, then the nodes of G have a legal coloring with r colors.
(2) If the nodes of G have a legal coloring with r colors, then the nodes of G′ have an s-free coloring with r colors.
(3) The number of nodes G′ can be upper bounded by a polynomial of the number of the nodes of G.

Thus for each legal (edge free) coloring problem we can construct a directed s-free coloring problem. If the second can be solved in polynomial time, it means that the first one can be solved in polynomial time as well.

Let $v_1, \ldots, v_n$ be the edges of G. In other words let $V = \{v_1, \ldots, v_n\}$. We consider an isomorphic copy $K_{i,j} = (W_{i,j}, F_{i,j})$ of the auxiliary graph K = (W, F) for each i, j, $1 \leq i < j \leq n$. We recall that the nodes of K are labeled by the numbers 1, . . . , k. The nodes of $K_{i,j}$ will be labeled by the ordered triples (i, j, 1), . . . , (i, j, k). Here the correspondence

$$1 \leftrightarrow \{i, j, 1), \ldots, l \leftrightarrow (i, j, k)$$

defines the isomorphism between K and $K_{i,j}$.

With each of the nodes $v_1, \ldots, v_n$ we associate a node $v'_1, \ldots, v'_n$ of the graph G′. At this moment our only concern is that $v'_1, \ldots, v'_n$ are pair-wise distinct points and they are nodes of G′. But G′ may have further nodes.
If the unordered pair $\{v_i, v_j\}$ is an edge of G, then we add additional k - 2 nodes to G′. We identify the nodes (i, j, k), (i, j, k) of $K_{i,j}$ with the nodes v'_i, v'_j of G′, respectively. Next, we add the remaining k - 2 nodes of $K_{i,j}$ to the nodes of G′. Finally, we add all the edges of $K_{i,j}$ to the edges of G′.

If the unordered pair $\{v_i, v_j\}$ is not an edge of G, then we do not add any nodes and we do not add any edges to G′. Clearly, G′ has directed edges and it has $|V| + |E| (k – 2)$ nodes. Since r, s are fixed numbers, it follows that k - 2 = c is a constant and so $|V|$ can be upper bounded by $n + cn(n - 1)/2$ which is a second degree polynomial in terms of n. This observation shows that condition (3) is satisfied.

In order to show that condition (1) is satisfied let us assume that $f' : V \to \{1, \ldots, r\}$ is a legal free coloring of the nodes of G′. Using f′ we define a coloring $f : V \to \{1, \ldots, r\}$ of the nodes of G. We set $f(v_i)$ to be equal to $f'(v'_i)$.

We claim that $f(v_i) = f(v_j)$ implies that the unordered pair $\{v_i, v_j\}$ is not an edge of G.

To verify the claim we assume on the contrary that $f(v_i) = f(v_j)$ and $\{v_i, v_j\}$ is an edge of G. The restriction of f′ to $W_{i,j}$ is an s-free coloring of the nodes of the graph $K_{i,j}$. By Lemma 5, the nodes (i, j, 1) and (i, j, k) cannot receive the same color. Using v'_i = (i, j, 1), v'_j = (i, j, k) we get

$f(v_i) = f'(v'_i) \neq f'(v'_j) = f(v_j)$,
a contradiction.

To demonstrate that condition (2) is satisfied let us suppose that $f : V \rightarrow \{1, \ldots, r\}$ be a legal coloring of the nodes of G. Using f we define a coloring $f' : V' \rightarrow \{1, \ldots, r\}$ of the nodes of G′. We set $f'(v'_i)$ to be equal to $f(v_i)$.

Let us consider two distinct nodes v'_i, v'_j of G′. If the unordered pair $\{v'_i, v'_j\}$ is an edge of G′, then by the construction of G′ the nodes v'_i, v'_j are identical with the nodes (i, j, 1), (i, j, k) of $K_{i,j}$, respectively. Thus v'_i = (i, j, 1), v'_j = (i, j, k). Since f is a legal coloring of the nodes of G, it follows that $f(v_i) \neq f(v_j)$ and so by the definition of f′, we get that $f'(v'_i) \neq f'(v'_j)$.

For the sake of definiteness let us suppose that $f'(v'_i) = 1$ and $f'(v'_j) = 2$. We have a partial coloring of the nodes of $K_{i,j}$. Namely, the nodes (i, j, 1), (i, j, k) are colored with colors 1, 2, respectively. Other nodes of $K_{i,j}$ are left uncolored. By Lemma 5, this partial coloring of $K_{i,j}$ can be extended to an s-free coloring of $K_{i,j}$. Since this can be accomplished in connection with each adjacent nodes v'_i, v'_j of G′, it follows that the nodes of G′ have an s-free coloring with r colors.

Second Proof: We present an informal new proof where the graph H plays a more direct role. The node edge incidence matrix M of a finite simple graph G = (V, E) is a |V| by |E| matrix. The rows and the columns of M are labeled by the nodes and the edges of G, respectively. If e = {u, v} is an edge of G, then we place two bullets into M. We put one bullet into the cell at the intersection of row u and column e. Then we put a bullet into the cell at the intersection of row v and column e.

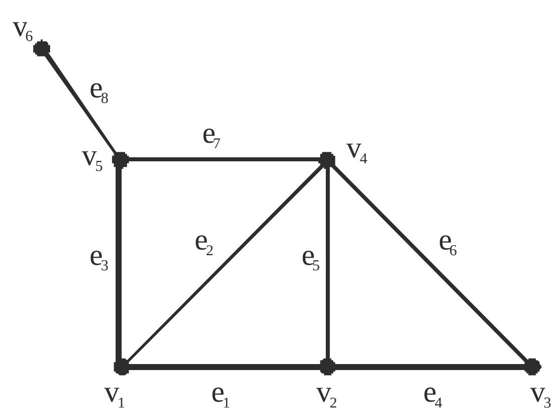

Figure 2: The toy example Γ

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	
v_1	•	•	•					
v_2	•			•	•			
v_3				•		•		
v_4		•			•	•	•	
v_5			•				•	•
v_6								•

Table 2: The node edge incidence matrix of the toy example Γ

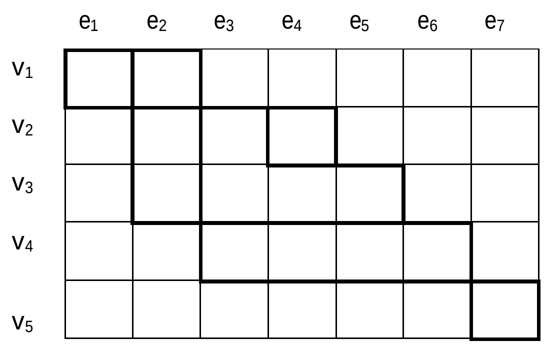

Figure 3: The graph associated with the toy example Γ

For the sake of illustration we included a toy example. The graph can be seen in Figure 2. The node edge incidence matrix of is in Table 2.

Suppose we are given a finite simple undirected graph G = (V, E). Using G we construct a graph G = (V, E). The construction is guided by the node edge incidence matrix of G. Let V = $\{v_1, . . . , v_n\}$, E = $\{e_1, . . . , e_m\}$. If $e_t = \{v_i, v_j\}$, then we add the ordered pairs (i, t), (j, t) to the set of nodes of G. Thus V is a set whose elements are ordered pairs. Clearly | V | = 2|E| = 2m.

We can form a mesh consisting of n horizontal and m vertical lines. The intersection of the horizontal and vertical lines form (n)(m) mesh points. The nodes of G can be identified with some of these mesh points.

Two distinct nodes (i, t) and (j, t) of G on a vertical mesh line are connected with a vertical undirected edge in G. Two distinct nodes (i, x) and (i, z) of G on a horizontal line are connected with a horizontal undirected edge in G if there is no node in the form (i, y) such that $x < y < z$. Figure 3 depicts the graph associated with the toy example. The mesh lines are represented by thin lines. Bold lines represent the edges of Γ^*.

We replace each horizontal edge of G by an isomorphic copy of the auxiliary graph H. Next we replace each vertical edge of G by an isomorphic copy of the auxiliary graph K. After all possible replacements we get a finite simple directed graph G′.

Suppose that the nodes of G′ have an s-free coloring with r colors. The isomorphic copies of the auxiliary graph H guarantee that the nodes of G′ on a fixed horizontal line all receive the same color.

The isomorphic copies of the auxiliary graph K make sure that the two nodes of G′ on a fixed vertical line receive distinct colors. In this way we get a legal coloring of the nodes of G with r colors.

5. CONCLUSION

Hence we suppose that the nodes of G have a legal coloring with r colors. This coloring will provide partial colorings of the nodes of the isomorphic copies of the graphs H and K. One can extend these partial colorings to a complete s-free coloring of the nodes of G′ with r colors.

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