Original Article

Bulletin of Pure and Applied Sciences.
Vol. 37E (Math & Stat.), No.1, 2018.P.94-100
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2018.00010.3

Numerical Treatment of Fisher's Equation using Finite Difference Method

M.L. Dhumal[1,*], S.B. Kiwne[2]

Author Affiliation:
[1]Dept. of Mathematics, Deogiri College, Aurangabad, Maharashtra, India-431005.
E-mail: mldhumal19@gmail.com
[2]Dept. of Mathematics, Deogiri College, Aurangabad, Maharashtra, India 431005.
E-mail: sbkiwne@yahoo.co.in

***Corresponding Author:**
Mrs. M.L. Dhumal, Assistant Professor, Department of Mathematics, Deogiri College, Station Road, Aurangabad, Maharashtra, India 431005

E-mail: mldhumal19@gmail.com

Received on 10.03.2018, **Accepted on 14.04.2018**

Abstract

A numerical method is proposed to approximate the solutions of nonlinear Fisher reaction diffusion equation using finite difference method. Fisher's equation describes the process of interaction between diffusion and reaction. The method is based on replacing each term in the Fisher's equation using finite difference method. Fisher equation has many applications in science and engineering fields. In this paper we study Du-Fort Frankel finite difference method for Fisher's equation. This method is derived from Leapfrog explicit scheme.

Keywords: Fisher's equation, Leapfrog explicit Scheme, Taylor series expansion
Subject Classification: AMS 65N06, 65N12, 65N22

1. INTRODUCTION

The Fisher equation arises in heat and mass transfer, biology and ecology. Fisher's equation has many applications such as gene propagation [1], neurophysiology [6] and tissue engineering [7]. In 1937 Fisher and Kolmogorov et al. investigated Fisher- Kolmogorov-Petrovsky-Piscounov (Fisher-KPP) [1], [3] equation which is widely known as Fisher's Equation.

The proposed nonlinear reaction-diffusion equation is defined as

$$u_t = Du_{xx} + SF(u)$$
$$u(x,0) = u_0 \qquad (0 \le x \le 1)$$

$$u(0,t) = f_1(t) \qquad (0 \le t \le \tau)$$

$$u(1,t) = f_2(t) \qquad (0 \le t \le \tau)$$

where D is diffusion coefficient and S is reactive factor, t is time, x is distance and F(u) is nonlinear function. This is a parabolic, nonlinear and non-homogeneous partial differential equation in two dimensions. In this equation, t plays the role of a time variable and x is a spatial variable; we are going to solve the partial differential equation with boundary and initial conditions in order to obtain a numerical solution.

In the past several decades, there has been great activity in developing numerical and analytical methods for the Fisher's equation. Explicit solution of the Fisher's equation for a special wave speed has been found by Ablowitz and Zeppetella [2]. Adomian's decomposition method has been applied to approximate the solutions of the proposed equation in [4], [6]. A numerical scheme for solving the Fisher's equation, which permits the usage of very large discretization mesh sizes in space and time, has been proposed in [4]. Further recently Fisher equation is studied by semi implicit scheme [11].

In this paper we study a numerical finite difference method named Du-Fort Frankel Method for non linear Fisher Equation. We apply this method for nonlinear function F(u) =u(1-u) which becomes a famous Fisher's equation. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population. We shall see from the numerical results that the proposed method is more accurate in comparatively less time than the method presented in [11].

The paper is organized as follows: in section 2 we discuss Du-Fort Frankel Method to solve Fisher Equation, in section 2.1 we give an illustrative example of Fisher equation. Numerical results that illustrate the efficiency of the proposed method are reported in Section 2.2. In Section 3 we present the stability of Du-Fort Frankel method. Section 3.1 contains Error Analysis of the method. Finally, in Section 4, we give the conclusion for this paper.

2. DU-FORT FRANKEL METHOD

In this section we introduce an explicit finite difference method known as Du-Fort Frankel method to solve Fisher's equation. This scheme is derived from Leapfrog explicit scheme. In this method we use central difference approximation in time and space derivatives.

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}, \frac{\partial u}{\partial x} - \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\,\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{(\Delta x)^2}$$

Fisher equation is given by,

$$u_t = Du_{xx} + su(1-u) \tag{1}$$

$$u(x,0) = u_0 \qquad (0 \le x \le 1) \tag{2}$$

$$u(0,t) = f_1(t) \qquad (0 \le t \le \tau) \tag{3}$$

$$u(1,t) = f_2(t) \qquad (0 \le t \le \tau) \tag{4}$$

Equation (1) contains non-linear term, we use method of lagging. In method of lagging one is calculated at n time level and other is calculated at n+1 time level. Equations (2),(3) and (4) represent initial and boundary condition. The solution domain is discretized with uniform meshes. The space interval is divided into N equal subintervals and time interval is divided into M equal subintervals. Using central difference approximation in time and space derivatives equation (1) becomes,

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} = D\,\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{(\Delta x)^2} + s u_j^{n+1}(1-u_j^{n}) \qquad (5)$$

Equation (5) is Leap-Frog explicit finite difference scheme. In this method if we replace middle term u_j^n by arithmetic mean of u_j^{n-1} and u_j^{n+1}, we get Du-Fort Frankel Method as follows:

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} - D\,\frac{u_{j+1}^{n}-u_j^{n+1}-u_j^{n-1}+u_{j-1}^{n}}{(\Delta x)^2} + s u_j^{n+1}(1-u_j^{n}) \qquad (6)$$

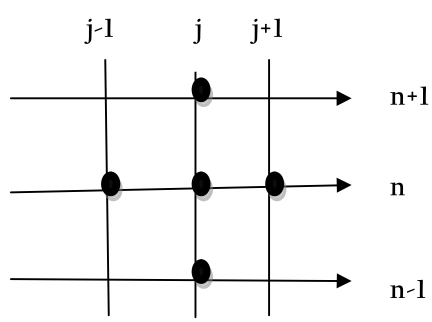
$$u_j^{n+1} = u_j^{n-1} + \frac{2D\Delta t}{(\Delta x)^2}\left[u_{j+1}^{n} - u_j^{n+1} - u_j^{n-1} + u_{j-1}^{n}\right] + 2s\Delta t u_j^{n+1}(1-u_j^{n})$$

Let $\frac{D\Delta t}{(\Delta x)^2} = p\,, s\Delta t = q$

$$u_j^{n+1} = u_j^{n-1} + 2p\left[u_{j+1}^{n} - u_j^{n+1} - u_j^{n-1} + u_{j-1}^{n}\right] + 2q u_j^{n+1}(1-u_j^{n})$$

$$(1+2p)\, u_j^{n+1} = (1-2p)\, u_j^{n-1} + 2p\left[u_{j+1}^{n} + u_{j-1}^{n}\right] + 2q u_j^{n+1}(1-u_j^{n}) \qquad (7)$$

This is required Du-Fort Frankel Scheme. It is two step method requiring value of u at time step n-1 and n to determine the value at current step n+1. But on the other hand, it requires the solution at the first time level to be determined by any other two time level scheme. For j = 2 onwards, this scheme can be applied. Here we applied backward difference approximations to find the solutions at first time level.



Symbolic representation of Du-Fort Frankel Scheme

2.1 Illustrative Example

Numerical solution of nonlinear Fisher equation is obtained by Du-Fort Frankel Method represented by finite difference scheme in formula (7).

Fisher equation with initial condition and homogeneous boundary conditions are as follows:

$$u_t = u_{xx} + 6u(1-u) \qquad (0 < x < 1), t > 0$$

$$u(x,0) = \frac{1}{(1+e^{x})^2} \qquad 0 \le x \le 1$$
$$u(0,t) = \frac{1}{(1+e^{5t})^2} \qquad 0 \le t \le \tau$$

$$u(1,t) = \frac{1}{(1+e^{1-5t})^2} \qquad 0 \le t \le \tau$$

Here D =1, S =6, $\Delta x = 0.25 \;\&\; \Delta t = 0.001$

$p = \frac{D\Delta t}{(\Delta x)^2} = \frac{1*0.001}{(0.25)^2} = 0.016$, $q = s\Delta t = 6 * 0.001$ =0.006

(1+2p) u_j^{n+1}= (1-2p) u_j^{n-1} + 2$p\left[u_{j+1}^{n} + u_{j-1}^{n}\right] + 2qu_j^{n+1}(1-u_j^{n})$

Substituting value of p and q, we get

1.032u_j^{n+1}= 0.968 u_j^{n-1} + 0.032$\left[u_{j+1}^{n} + u_{j-1}^{n}\right] + 0.012u_j^{n+1}(1-u_j^{n})$

Taking approximations we get the following result.

2.2 Numerical Results:

Numerical results are computed using the Du-Fort Frankel scheme in order to validate our proposed scheme with exact solution of the Fisher equation. This scheme is more accurate and gives the result in less time as compared to semi implicit scheme given in [11].

Table 1: Numerical result by Du-Fort Frankel Method with $\Delta x = 0.25 \;\&\; \Delta t = 0.001$

Value of x	Numerical solution by Du-Fort Frankel Method	Exact Solution
0.00	0.2624	0.2626
0.25	0.2025	0.2026
0.50	0.1516	0.1516
0.75	0.1101	0.1100
1.00	0.0777	0.0777

3. STABILITY ANALYSIS

The stability of a numerical scheme is associated with propagation of numerical error. The stability of numerical schemes can be investigated by performing von Neumann stability analysis. The von Neumann stability method consists of substituting the trial function

$$u_j^{n+1} = Ae^{i\omega j}\xi^{n}$$

into the approximate difference scheme and then to find a characteristic equation for the amplification factor ξ.

Finding what restrictions on the parameters are required to have $|\xi| \le 1$: the scheme is von Neumann stable if $|\xi| \le 1$, and von Neumann unstable if $|\xi| > 1$. When the characteristic equation has multiple roots we require them to be distinct.

The Finite difference scheme of Du Fort-Frankel method is as follows-

$$u_j^{n+1} = u_j^{n-1} + 2p\left[u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right] + 2qu_j^{n+1}(1 - u_j^n)$$

which can be rearranged as

$$u_j^{n+1} = \frac{(1-2p)}{(1+2p)}u_j^{n-1} + \frac{2p}{(1+2p)}\left[u_{j+1}^n + u_{j-1}^n\right] + \frac{2q}{(1+2p)}\,u_j^{n+1}(1 - u_j^n)$$

To study the stability we use trial solution of the form $u_j^{n+1} = Ae^{i\omega j}\xi^n$, we get

$$Ae^{i\omega j}\xi^{n+1} = \frac{(1-2p)}{(1+2p)}Ae^{i\omega j}\xi^{n-1} + \frac{2p}{(1+2p)}\left[Ae^{i\omega_{j+1}}\xi^n + Ae^{i\omega_{j-1}}\xi^n\right] +$$
$$\frac{2q}{(1+2p)}Ae^{i\omega j}\xi^{n+1}(1 - Ae^{i\omega j}\xi^n)$$

Divide the above expression by $Ae^{i\omega j}\xi^{n-1}$, it gives

$$\xi^2 = \frac{(1-2p)}{(1+2p)} + \frac{2p}{(1+2p)}\left[e^{i\omega} + e^{-i\omega}\right] + \frac{2q}{(1+2p)}(1 - Ae^{i\omega j}\xi^n)\xi^2$$

Using the relation $\left[e^{i\omega} + e^{-i\omega}\right] = 2\cos\omega$, we obtain

$$(1+2p)\left\{1 - 2q\left(1 - Ae^{i\omega j}\xi^n\right)\right\}\xi^2 - 4p\xi\cos\omega - 1 = 0 \qquad (8)$$

Stability requires $|\xi| \le 1$ and that the roots ξ are all distinct. The two roots of the above equation are given by

$$\xi_\pm - \frac{1}{2(1+2p)\{1-2q(1-Ae^{i\omega j}\xi^n)\}}\left[\frac{4p\cos\omega \pm \sqrt{16p^2\cos^2\omega + 4(1-4p^2)}}{\{1-2q\left(1-Ae^{i\omega j}\xi^n\right)\}}\right]$$

To avoid some complications we rewrite this equation as follows

$$\xi_\pm = \frac{1}{(1+2p)}\left[2p\cos\omega \pm \sqrt{4p^2\cos^2\omega + 1 - 4p^2}\right]$$

$$\xi_\pm = \frac{1}{(1+2p)}\left[2p\cos\omega \pm \sqrt{1 - 4p^2\sin^2\omega}\right] \qquad (9)$$

Case I: If $1 - 4p^2\sin^2\omega < 0$ then ξ_+ and ξ_- are complex roots, with $\xi_- = \overline{\xi_+}$, Moreover

$$|\xi_-|^2 = |\xi_+|^2 = \frac{4p^2\cos^2\omega + 4p^2\sin^2\omega + 1}{(1+2p)^2} = \frac{4p^2+1}{(1+2p)^2} \qquad (10)$$

and $0 < \frac{4p^2+1}{(1+2p)^2} < 1$, therefore $|\xi_-| \le 1$ and $|\xi_+| \le 1$.

Case II: If $1 - 4p^2\sin^2\omega > 0$

$$\frac{1}{(1+2p)}\left[2p\cos\omega + \sqrt{1 - 4p^2\sin^2\omega}\right] \le \frac{2p\cos\omega + 1}{1+2p} \le 1 \qquad (11)$$

Therefore $\xi_+ \leq 1$, where we used the fact that $0 < 1 - 4p^2 sin^2\omega \leq 1$.
Moreover

$$\frac{1}{(1+2p)}\left[2p\cos\omega + \sqrt{1-4p^2\sin^2\omega}\right] \geq \frac{-2p}{1+2p} \geq -1 \qquad (12)$$

Therefore $\xi_+ \geq -1$, where we have used the fact that $0 < 1 - 4p^2 sin^2\omega$ and $cos\ \omega \geq -1$.
We apply the same analysis for ξ_-, we have

$$\frac{1}{(1+2p)}\left[2p\cos\omega - \sqrt{1-4p^2\sin^2\omega}\right] \leq \frac{2p}{1+2p} \leq 1 \qquad (13)$$

Therefore, $\xi_- \leq 1$.

Moreover

$$\frac{1}{(1+2p)}\left[2p\cos\omega - \sqrt{1-4p^2\sin^2\omega}\right] \geq \frac{-2p-1}{1+2p} \geq -1 \qquad (14)$$

Therefore $\xi_- \geq -1$.

This analysis shows that if the condition $|\xi| \leq 1$ is satisfied then this method becomes unconditionally stable.

3.1 Error Analysis:

In this section we establish error analysis. Using Taylor's series expansion we expand the terms u_j^{n+1}, u_j^{n-1}, u_{j+1}^n, u_{j-1}^n as follows:

$$u_j^{n+1} = u_j^n + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt} + \frac{(\Delta t)^3}{3!} u_{ttt} + O(\Delta t)^4 \qquad (15)$$

$$u_j^{n-1} = u_j^n - \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt} - \frac{(\Delta t)^3}{3!} u_{ttt} + O(\Delta t)^4 \qquad (16)$$

$$u_{j+1}^n = u_j^n + \Delta x u_x + \frac{(\Delta x)^2}{2!} u_{xx} + \frac{(\Delta x)^3}{3!} u_{xxx} + O(\Delta x)^4 \qquad (17)$$

$$u_{j-1}^n = u_j^n - \Delta x u_x + \frac{(\Delta x)^2}{2!} u_{xx} - \frac{(\Delta x)^3}{3!} u_{xxx} + O(\Delta x)^4 \qquad (18)$$

Now substituting the value of equation (15), (16), (17) and (18) in equation (6)

The truncation error is given by

$$T_j^{n+1} = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - D\frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{(\Delta x)^2} - s u_j^{n+1}\left(1 - u_j^n\right) \qquad (19)$$

We get the error

$$T_j^{n+1} = u_t - Du_{xx} - su(1-u) + (\Delta t)^2[u_{ttt} + o((\Delta t)^2 + - - -] - D(\Delta x)^2[u_{xxxx} + o(\Delta x)^2 + - - -]$$

+--

The partial derivatives of u with respect to x and t are derived at point (x_j,t_n). Thus the truncation error T_j^{n+1} is of $o(\Delta x)^2, o(\Delta t)^2$. As we have used centre time and centre space approximation therefore the scheme is of order two in $\Delta x, \Delta t$. The difference scheme is convergent.

4. CONCLUSION

In this paper, the solution of the Fisher's equation is successfully obtained by Du-Fort Frankel Scheme. We have proved that though it is an explicit method but it is an unconditionally stable method. Also we have shown that difference scheme is convergent and is of order two. It is observed that the method is more accurate than the existing numerical methods [11], [12].

5. REFERENCES

[1]. R. A. Fisher, The wave of advance of advantageous genes. Ann. Eugenics 7(4) (1937) 355-369.

[2]. M.J. Ablowitz and A. Zeppetella, Explicit solution of Fisher's equation for a special wave. Speed, Bull Math. Biol., 41 (1979) 835-840.

[3]. A. Kolmogorov, I. Petrovskii and N. Piskunov, Study of the diffusion equation with growth of the quantity of matter and its application to biology problems, Moscow Univ. Bull. Math., 1 (1937) 1-25.

[4]. T. Mavoungou and Y. Cherruault, Numerical study of Fisher's equation by Adomian's method, Math. Comput. Modelling, 19 (1994) 89-95

[5]. N. Parekh and S. Puri, A new numerical scheme for the Fisher equation, J. Phys. And Math. Gen., 23 (1990) 1085-1091.

[6]. A.M. Wazwaz and A. Gorguis, An analytic study of Fisher's equation by using Adomian decomposition method, Appl. Math. Comput. 154 (2004) 609-620

[7]. H.C. Tuckwell, Introduction to theoretical neurobiology, Cambridge, UK: Cambridge University Press, 1988.

[8]. P.K. Maini, D.L.S. McElwain, D. Leavesley, Travelling waves in a wound healing assay, Appl. Math. Lett. 17 (2004) 575-580

[9]. A.R Mitchell and D.F. Griffiths., Handbook of Finite difference method in partial differential Equations, Wiley-Interscience Publication, 1980

[10]. Polyanin, A.D. and Zaitsev, V.F., Handbook of Nonlinear partial Differential Equations, Champan and Hall/CRC, Boca Raton, 2004.

[11]. Vinay Chandraker, Ashish Awasthi and Simon Jayaraj, A Numerical treatment of Fisher Equation, procedia engineering 127(2015) pp 1256-1262

[12]. Mehdi Bastani, Davod Khojasteh Salkuyeh, A highly accurate method to solve Fisher's equation, Pramana –Journal of Physics, Vol.78, Issue 3 (2012) pp 335-346

[13]. Sharefa Eisa Ali Alhazmi, Numerical solution of Fisher's equation using finite difference, Bulletin of Mathematical Sciences and Applications, Vol. 12, pp 27-34

[14]. J. D. Murray. Mathematical Biology. Springer-Verlag, New York, 1993

[15]. Benoit Cushman-Rosin, Analytical, Linear stability Criteria for Leap-Frog, Dufort-Frankel Method, Journal of Computational Physics, Academic Press, New York and London, Vol 53, No.2, February 1984.