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On Harmonious Graphs

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Abstract

Let G = (V (G), E(G)) be a graph with q edges. A function f is called harmonious labeling of graph G if f:V→{0,1,2,...,q-1} is injective and the induced function f* : E → {0,1,2,...,q} defined as f*(uv) = (f(u) + f(v))(mod q) is bijective. A graph which admits harmonious labeling is called harmonious graph. In this paper we prove that the jewel graph, triangular ladder graph, special flower graph, duplicating all the vertex of mK_1, in P_2+mK_1, $T(P_n) \odot K_m^c$ are harmonious graphs.

Keywords: Labeling, Jewel graph, Triangular ladder, Helm, Special flower graph, Harmonious labeling.

1. INTRODUCTION

A labeling or valuation or numbering of a simple graph G is a one to one mapping from its vertex (edges) into a set of integers which induces an assignment of labels to the edges (vertices) of G. For all terminology and notations in graph theory we follow Harary [3]. In 1980, Graham and Sloane suggested a new labeling known as Harmonious labeling [6]. In this paper we shall mean a simple finite graph without isolated vertices. Let G be a graph with q edges is said to be Harmonious if there is an injection f from the vertex set of G to {0,1,2,...,q-1} such that each edge (xy) is assigned the label (f(x) + f(y))(modq), the resulting edge labels are distinct.

Definition: 1.1
The Jewel graph J_n is a graph with vertex set $V(J_n) = \{u,x,v,y,u_i / 1 \le i \le n\}$ and the edge set $E(J_n) = \{ux, vx, uy, vy, xy, uu_i, vu_i / 1 \le i \le n\}$

Theorem: 1.1
The Jewel graph J_n is harmonious for $n \geq 1$.

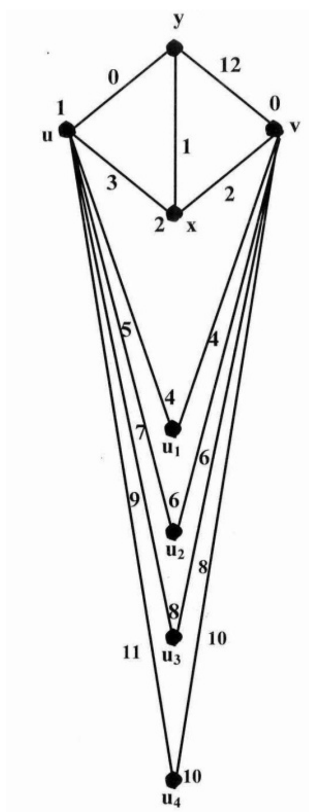
Proof

Let J_n be a (p,q) graph. Let the vertices set of Jewel graph is
$V(J_n) = \{u,x,v,y,u_i / 1\leq i \leq n\}$ and the edge set is $E(J_n) = \{ux,vx,uy,xy,uu_i,vu_i / 1\leq i \leq n\}$
Thus J_n has n+4 vertices and 2n+5 edges.
Define $f : V(J_n) \rightarrow \{0,1,2,\ldots,q\text{-}1)$ as follows
$f(u) = 1$
$f(v) = 0$
$f(y) = q-1$
$f(x) = 2$
$f(u_i) = 2i+2,\ 1\leq i \leq n$

Then the function f induces the function f* on $E(J_n)$ as follows.

$f^*(uy) = 0$
$f^*(xy) = 1$
$f^*(xv) = 2$
$f^*(ux) = 3$
$f^*(vy) = q-1$
$f^*(uu_i) = 2i+3,\ 1\leq i \leq n$
$f^*(vu_i) = 2i+2,\ 1 \leq i \leq n$
Thus the edge labels are distinct.
Hence J_n is Harmonious, for $n\geq1$

Illustration: 1.1
J_4 is Harmonious



Definition: 1.2

A triangular ladder TL_n, $n \geq 2$ is a graph obtained from a ladder L_n by adding the edges u_iv_{i+1} for $1 \leq i \leq n-1$, where u_i and v_i, $1 \leq i \leq n$ are the vertices of L_n, such that $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ are two paths of length n.

Theorem: 1.2

The triangular ladder TL_n is Harmonious for all n.

Proof:

Let TL_n be a (p,q) graph. Let u_i, v_i, $1 \leq i \leq n$ be the vertices of L_n. Join u_i to v_{i+1}, $1 \leq i \leq n-1$. The resultant graph is TL_n.

The edge set is $E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{u_iv_i / 1 \leq i \leq n\}$

Thus TL_n has 2n vertices and 4n-3 edges.

Define f: $V(TL_n) \rightarrow \{0,1,2,\ldots,q-1\}$as follows.

$f(u_i) = 2i$, $1 \leq i \leq n$

$f(v_i) = 2i-1$, $1 \leq i \leq n$.

Then the function f induces the function f* on $E(TL_n)$ as follows.

$f^*(u_nv_n) = 2$

$f^*(u_{n-1}u_n) = 1$

$f^*(u_{n-1}v_n) = 0$

$f^*(u_iu_{i+1}) = 4i+2$, $1 \leq i \leq n-2$

$f^*(v_iv_{i+1}) = 4i$, $1 \leq i \leq n-1$

$f^*(u_iv_{i+1}) = 4i+1$, $1 \leq i \leq n-2$

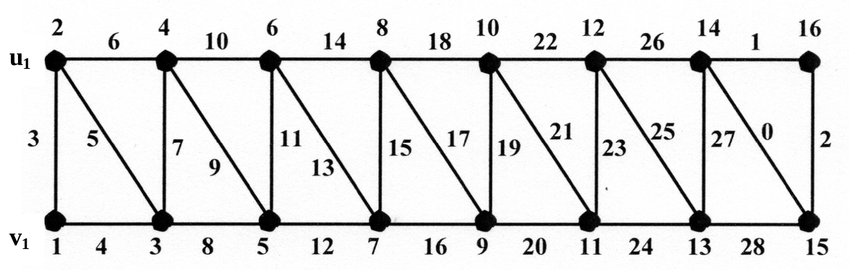
$f^*(u_iv_i) = 4i-1$, $1 \leq i \leq n-1$

Clearly all the edge labels are distinct.

$\therefore$ TL_n is Harmonious.

Illustration: 1.2

TL_8 is Harmonious



Definition: 1.3

A special flower ($F\ell_n$) is a graph obtained from a Helm H_n by joining each pendant vertex to the central vertex of the helm.

Theorem: 1.3

The special Flower graph $F\ell_n$ is Harmonious for all odd n.

Proof:

Let w be the central vertex of H_n and $v_1,v_2,...,v_n$ be the rim vertices of the cycle. Let $v_1',v_2',...,v_n'$ be the pendant vertices attached with v_i, i = 1,2,...,n

Now $v(F\ell_n) = \{w,v_1,v_2,...,v_n,v_1',v_2',...,v_n'\}$ and
$E(F\ell_n) = \{wv_i / 1\le i \le n\} \cup \{wv_i' / 1\le i \le n\} \cup \{v_iv_i' / 1\le i \le n\} \cup \{v_iv_{i+1} / 1\le i \le n$, where $v_{n+1}=v_1\}$

Note that $F\ell_n$ has 2n+1 vertices and 4n edges.

Define $f:V(G) \rightarrow \{0,1,2,...,q-1\}$ as follows

$f(w) = 0$

$f(v_i) = 2i-1,\ 1\le i \le n$

$f(v_2') = f(v_n)+2$

$f(v_{i+1}') = f(v_i')+2,\ 2\le i \le n$, where $v_{n+1}' = v_1'$

Then the function f induces the function f* on $E(F\ell_n)$ as follows.

$f^*(wv_i) = v_i,\ 1\le i \le n$

$f^*(wv_i') = v_i',\ 1\le i \le n$

$f^*(v_iv_{i+1}) = 4i,\ 1\le i \le n-1$

$f^*(v_nv_1) = 2n$

$f^*(v_1v_1') = 0$

$f^*(v_2v_2') = 2n+4$

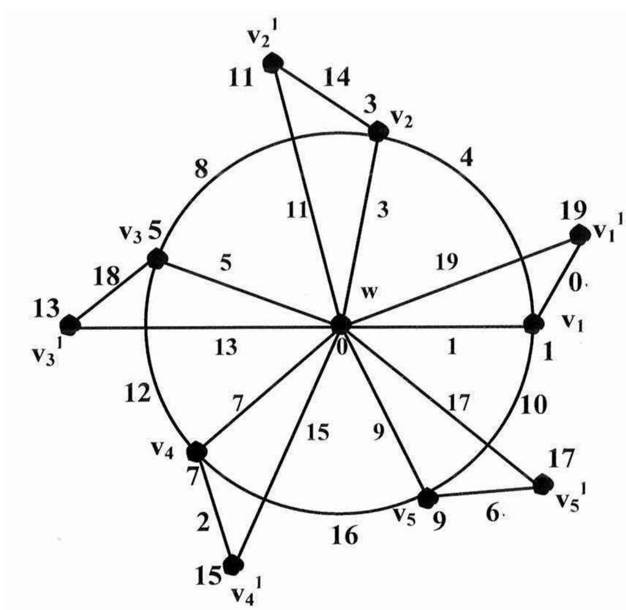
$f^*(v_iv_i') = [f(v_{i-1}v_{i-1}')+4](mod q),\ 3\le i \le n$

Clearly all the edge labels are distinct.

$\therefore F\ell_n$ is Harmonious for all odd n.

Illustration: 1.3

$F\ell_5$ is Harmonious



Theorem: 1.4

The graph G* obtained by duplicating all the vertex of mK_1 in P_2+ mK_1 is a Harmonious graph.

Proof:

The vertex set is $V(P_2+mK_1) = \{v_1, v_2, y_1, y_2, y_3, ..., y_m\}$, where $V(P_2) = \{v_1,v_2\}$ and $V(mK_1) = \{y_1,y_2,...,y_m\}$

Now duplicating all the vertex of mK_1 in P_2+mK_1

The resulting graph is G^* and whose vertex set is
$V(G^*) = \{v_1, v_2, y_1, y_2, \ldots, y_m, y_1', y_2', \ldots, y_m'\}$

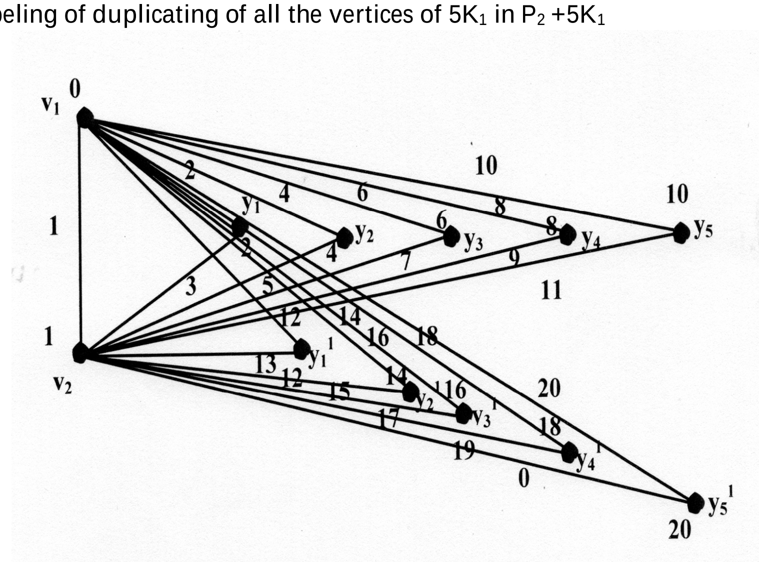
The edge set of G^* is
$E(G^*) = \{(v_1v_2) \cup (v_1y_i) / 1 \le i \le m\} \cup \{(v_1y_i') / 1 \le i \le m\} \cup \{(v_2y_i) / 1 \le i \le m\} \cup \{(v_2y_i') / 1 \le i \le m\}$
Note that G^* has $2(m+1)$ vertices and $4m+1$ edges.

Define $f: V(G) \to \{0,1,2,\ldots,q-1\}$ as follows.
$f(v_1) = 0$
$f(v_2) = 1$
$f(y_i) = 2i, 1 \le i \le m$
$f(y_i') = 2(m+i), 1 \le i \le m$

Then the function f induces the function f^* on $E(G^*)$ as follows.
$f^*(v_1y_i) = 2i, 1 \le i \le m$
$f^*(v_2y_i) = 2i+1, 1 \le i \le m$
$f^*(v_1y_1') = f(v_1y_m)+2$
$f^*(v_1y_{i+1}') = f(v_1y_i')+2, 1 \le i \le m-1$
$f^*(v_2y_1') = f(v_2y_m) + 2$
$f^*(v_2y_{i+1}') = f(v_2y_i') + 2, 1 \le i \le m-1$
$f^*(v_1v_2) = 1$

Here all the edge labels are distinct.
∴ The graph G^* is Harmonious.

Illustration: 1.4
Harmonious labeling of duplicating of all the vertices of $5K_1$ in $P_2 + 5K_1$



Theorem: 1.5
The graph $G = T(P_n) \odot K_m^c$ is a Harmonious graph for all m and n

Proof:
The vertex set of G is $V(G) = \{u_i / 1 \le i \le n\} \cup \{v_i / 1 \le i \le n-1\} \cup \{u_{ij} / 1 \le i \le n, 1 \le j \le m\} \cup \{v_{ij} / 1 \le i \le n-1, 1 \le j \le m\}$

The edge set of G is
$E(G) = \{(u_iu_{i+1}) / 1 \le i \le n-1\} \cup \{(v_iv_{i+1}) / 1 \le i \le n-2\} \cup \{(u_iv_i) / 1 \le i \le n-1\} \cup \{(v_iu_{i+1}) / 1 \le i \le n-1\} \cup \{(u_iu_{ij}) / 1 \le j \le m, 1 \le i \le n\} \cup \{(v_iv_{ij}) / 1 \le i \le n-1, 1 \le j \le m\}$

Then $V(G) = 2n(m+1) - m-1$

$E(G) = 2n(m+2)-m-5$

Define $f:V(G) \rightarrow \{0,1,2,\ldots,q-1\}$ as follows

$f(v_i) = 2i,\ 1\leq i \leq n-1$

$f(u_i) = 2i-1,\ 1\leq i \leq n$

$f(v_{(n-1)1}) = f(v_{n-1}) + n$

$f(v_{(-i+n-1)1}) = f(v_{(-i+n)1}) +1,\ 1\leq i \leq n-2$

$f(v_{(-i+n)j}) = f(v_{(-i+n)(j-1)})+n-1, 1\leq i \leq n-1, 2\leq j \leq m$

$f(u_{n1}) = f(V_{1m})+1$

$f(u_{(-i+n)1}) = f(u_{(-i+n+1)1})+1,\ 1 \leq i \leq n-1$

$f(u_{(-i+n+1)(j+1)} = f(u_{(-i+n+1)j})+n, 1\leq i\leq n, 1\leq j \leq m-1$

Then the function f induces the function f* on E(G) as follows.

$f^*(v_iv_{i+1}) = 4i+2,\ 1\leq i \leq n-2$

$f^*(u_iu_{i+1}) = 4i,\ 1\leq i \leq n-1$

$f^*(u_iv_i) = 4i-1,\ 1\leq i \leq n-1$

$f^*(u_{i+1}v_i) = 4i+1,\ 1 \leq i \leq n-1$

$f^*(v_1v_{11}) = f(v_{n-1}u_n)+1$

$f^*(v_{i+1}v_{(i+1)1}) = f(v_iv_{i1}) + 1,\ 1\leq i \leq n-2$

$f^*(v_iv_{i(j+1)} = f(v_iv_{ij})+(n-1), 1\leq i \leq n-1, 1\leq j \leq m-1$

$f^*(u_1u_{11}) = f(v_{n-1}v_{(n-1)m})+1$

$f^*(u_{i+1}u_{(i+1)1}) = f(u_iu_{i1})+1,\ 1\leq i \leq n-1$

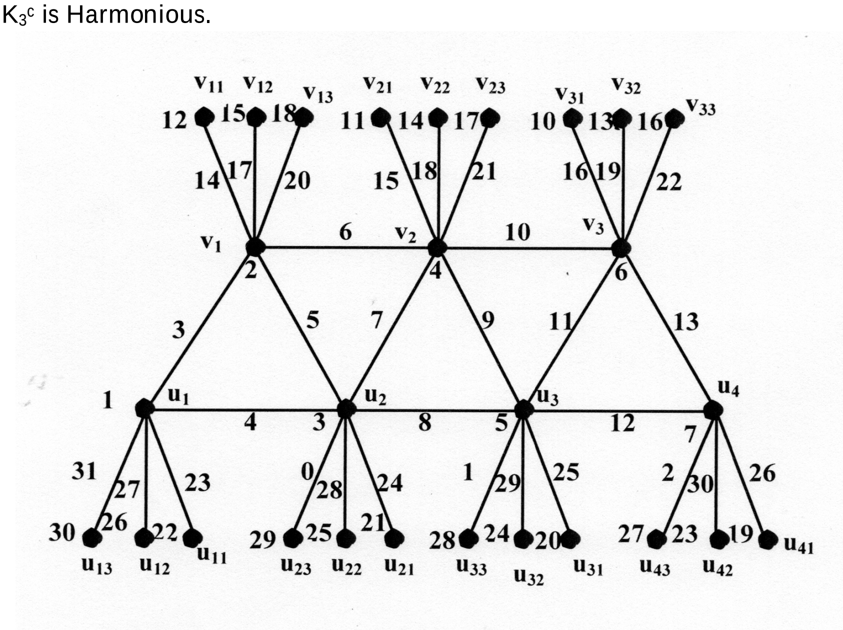
$f^*(u_iu_{i(j+1)}) = (f(u_iu_{ij})+n) \bmod q, 1\leq i \leq n, 1\leq j \leq m-1$

Hence the edge labels are distinct.

$\therefore$ The graph is Harmonious.

Illustration: 1.5

$T_4 \odot K_3^c$ is Harmonious.



REFERENCES

[1]. Dr. Devaraj .J and Teffilia. M, (2016). On Ek-cordial graphs, IJAIR (2278-7844)#85 / vol 5, Issue 11, November 2016.

[2]. Dr. Devaraj. J, Teffilia. M, On Difference cordial graphs (accepted, IJMSEA)

[3]. Harary F, (1993). Graph theory, Narosa Publishing house, New Delhi.

[4]. Jospeh. A. Gallian, (2015). Dynamic Survey of graph labeling.

[5]. Renuka and others (2013). On Harmonious labeling, American Journal of Mathematical Science and Applications, vol 1, No. 1, January-June 2013, PP: 55-60.

[6]. M. Seenivasan and A.Lourdusamy, Absolutely Harmonious labeling of graphs, International J. Math. Combin. Vol 2 (2011) 40-51.