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Common Fixed Point for Compatible Mappings of Type (α) Satisfying an Implicit Relation

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Abstract

Here we prove a common fixed point theorem for compatible mappings of type (α) satisfying an implicit relation. We extend results of Popa [9] for five mappings.

Keywords: Implicit relation, compatible mappings, complete metric space, fixed point

1. INTRODUCTION

In 1976, Jungck [3] proved a common fixed point theorem for commuting mappings generalizing the Banach's fixed point theorem. Sessa [10] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [4] introduced a more generalized notion of commutativity, the so called compatibility, which has a more general character than that of the weak commutativity. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [8]. Also Jungck [4] extended the results of Khan and Imdad [7] and proved common fixed point theorems for four mappings by using one of the mappings continuity and employing conditions of compatible mappings. Kang, Cho and Jungck [6] extended the results of Ding [1], Diviccaso and Sessa [2] and proved common fixed point theorems. Recently Sharma [11], Sharma and Patidar [15], Sharma and Deshpande [13, 14], Sharma and Choubey [12] have worked on this line. In 1993, Jungck, Murthy and Cho [5] introduced the concept of compatible mappings of type (α) in metric spaces. Now we begin with some definitions.

Definition 1.1: Let (X , d) be a metric space.

(1) A sequence $\{x_n\}$ in a metric space (X , d) is said to be convergent to a point x in X, if $\lim_{n\to\infty} d(x_n , x) = 0$.

(2) A sequence $\{x_n\}$ in a metric space (X , d) is said to be Cauchy sequence , if $\lim_{m,n\to\infty} d(x_m , x_n) = 0$.

(3) A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.2: (Jungck [4]) Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be compatible, if

$\lim_{n\to\infty} d(ABx_n , BAx_n) = 0$,
where $\{x_n\}$ is a sequence in x such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 1.3: (Jungck et al. [5]) Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be compatible of type (α), if

$$\lim_{n\to\infty} d(ABx_n , BBx_n) = 0,$$

and
$\lim_{n\to\infty} d(BAx_n , AAx_n) = 0$,

where $\{x_n\}$ is a sequence in x such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Propostion 1.1: (Jungck [4]) Let (X,d) be a metric space. Let A and B be continuous mappings from X into itself. Then A and B are compatible if and only if they are compatible of type (α).

Propostion 1.2: (Jungck et al. [5]) Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B are compatible of type (α) and Az = Bz for some $z \in X$, then

$$ABz = BBz = BAz = AAz.$$

Propostion 1.3: (Jungck et al. [5]) Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B are compatible of type (α) and $\{x_n\}$ is a sequence in X such then

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, then

(i) $\lim_{n\to\infty} BAx_n = Az$ if A is continuous at z.
(ii) ABz = BAz and Az = Bz , if A and B are continuous at z.

Implicit Relations:

Let Ψ be the set of all real continuous functions $\Phi(t_1, t_2,\ldots\ldots,t_6) : R^6_+ \to R$ satisfying the following conditions:
(Φ_1) $\Phi(t_1, t_2,\ldots\ldots,t_6)$ is decreasing in the variables $t_2,\ldots\ldots,t_6$.
(Φ_2) there exists $h \in (0,1)$ such that for every $u,v \geq 0$,
(Φ_a) : $\Phi(u,v,v,u,0,u+v) \leq 0$ implies $u \leq hv$.
(Φ_b) : $\Phi(u,0,u,0,u,u) > 0\ \forall\ u > 0$, and
(Φ_c) : $\Phi(u,0,0,u,0,u) > 0\ \forall\ u > 0$.

Example 1.1: $\Phi(t_1, t_2,\ldots\ldots , t_6) = t_1 – k \max \{ t_2, t_3, t_4, ½ (t_5 + t_6) \}$ where $k \in (0 ,1)$
(Φ_1): Obviously true.

(Φ_a): Let $u > 0$ then
$\Phi(u,v,v,u,0,u+v) = u - k \max\{v, v, u, \frac{1}{2}(u+v)\} \leq 0$.

If $u \geq v$ then $u \leq ku < u$, a contraction. Thus $u < v$ and $u \leq kv = hv$, where $h = k \in (0,1)$.
If $u = 0$, then $u \leq hv$.
(Φ_b) : $\Phi(u,0,u,0,u,u) = u - k \max\{0, u, 0, \frac{1}{2}(u+u)\}$
$= (1-k)u > 0 \quad \forall u > 0$.
(Φ_c) : $\Phi(u,0,0,u,0,u) = u - k \max\{0, 0, u, \frac{1}{2}(0+u)\}$
$= (1-k)u > 0 \quad \forall u > 0$.

Example 1.2: $\Phi(t_1, t_2, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$
where $a > 0$, $b.d \geq 0$, $0 \leq c < 1$, $a + b + c < 1$ and $b + d < 1$.

(Φ_1): Obviously holds.
(Φ_a): Let $u > 0$ then
$\Phi(u,v,v,u,0,u+v) = u^2 - u(av + bv + cu) - d.0 \leq 0$.

If $u \leq (a+b)/(1-c)\, v = hv$, where $h = (a+b)/(1-c) \leq 1$.

Therefore, $u \leq hv$. If $u = 0$, then $u \leq hv$.
(Φ_b) : $\Phi(u,0,u,0,u,u) = u^2(1-(b+d)) > 0 \quad \forall u > 0$.
(Φ_c) : $\Phi(u,0,0,u,0,u) = u^2(1-c) > 0 \quad \forall u > 0$.

Example 1.3: $\Phi(t_1, t_2, \ldots, t_6) = t_1 - k[\max\{t_2^2, t_3t_4, t_5t_6, t_3t_5, \frac{1}{2}t_4t_6\}]^{1/2}$,
(Φ_1): Obviously
(Φ_a): Let $u > 0$ then
$\Phi(u,v,v,u,0,u+v) = u - k[\max\{v^2, vu, 0, 0, \frac{1}{2}u(u+v)\}]^{1/2} \leq 0$.

If $u \geq v$ then $u \leq ku < u$, a contraction. Thus $u < v$ and $u \leq kv = hv$, where $h = k \in (0,1)$.

If $u = 0$, then $u \leq hv$.
(Φ_b) : $\Phi(u,0,u,0,u,u) = (1-k)u > 0, \quad \forall u > 0$.
(Φ_c) : $\Phi(u,0,0,u,0,u) = (1-k/\sqrt{2})u > 0, \ \forall u > 0$.

Popa [9] proved the following.

Theorem A : Let (X,d) be a complete metric space and A,B, S and T be mappings from X into itself such that
(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(ii) One of A,B, S and T mappings is continuous,
(iii) the pairs {A,S} and {B,T} are compatible,
(iv) the inequality

$\Phi(d(Ax,By), d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)) \leq 0$
for all $x, y \in X$ and $\Phi \in \Psi$. Then A,B, S and T have a unique common fixed point in X.

We extend Theorem A for five mappings, moreover a different proof is given.

Theorem 1.1: Let (X,d) be a complete metric space and A,B, S, T and P be mappings from X into itself such that
(1.1) $P(X) \subset AB(X)$ and $P(X) \subset ST(X)$,
(1.2) PB = BP, PT = TP, AB = BA, ST = TS,
(1.3) A and B are continuous,
(1.4) the pair {P,AB} is compatible of type (α),
(1.5) $d(x,STx) \geq d(x,ABx)$, for all $x \in X$,

(1.6) the inequality

$\Phi(d(Px,Py), d(ABx,Px), d(ABx,STy), d(STy,Py), d(STy,Px), d(ABx,Py)) \leq 0$
for all x, y $\in$ X and $\Phi \in \Psi$.

Then A, B, S , T and P have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Since $P(X) \subset AB(X)$, we choose a point $x_1 \in X$ such that $Px_0 = ABx_1$ and since $P(X) \subset ST(X)$, for this a point x_1, there exists $x_2 \in X$ such that $Px_1 = STx_2$ .Inductively, we can define a sequence $\{y_n\}$ in X such that
$y_{2n} = Px_{2n} = ABx_{2n+1}$ and
$y_{2n+1} = Px_{2n+1} = STx_{2n+2}$, n = 0,1,2,…..

Letting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (1.6) , we write
$\Phi(d(Px_{2n+1} ,Px_{2n+2}), d(ABx_{2n+1},Px_{2n+1}), d(ABx_{2n+1},STx_{2n+2}), d(STx_{2n+2},Px_{2n+2}),$

$d(STx_{2n+2},Px_{2n+1}), d(ABx_{2n+1},Px_{2n+2})) \leq 0.$

$\Phi(d(y_{2n+1} ,y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n+1},y_{2n+1}), d(y_{2n},y_{2n+2})) \leq 0.$

$\Phi(d(y_{2n+1} ,y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), 0, d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})) \leq 0.$

By condition (Φ_a), we have
(1.7) $d(y_{2n+1} ,y_{2n+2}) \leq h\, d(y_{2n} ,y_{2n+1})$

Similarly, by putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.6), we have
$\Phi(d(Px_{2n} ,Px_{2n+1}), d(ABx_{2n},Px_{2n}), d(ABx_{2n},STx_{2n+1}), d(STx_{2n+1},Px_{2n+1}),$

$d(STx_{2n+1},Px_{2n}), d(ABx_{2n},Px_{2n+1})) \leq 0.$

$\Phi(d(y_{2n} ,y_{2n+1}), d(y_{2n-1},y_{2n}), d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), d(y_{2n},y_{2n}), d(y_{2n-1},y_{2n+1})) \leq 0.$

$\Phi(d(y_{2n} ,y_{2n+1}), d(y_{2n-1},y_{2n}), d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), 0, d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})) \leq 0.$

By condition (Φ_b), we have
(1.8) $d(y_{2n} ,y_{2n+1}) \leq h\, d(y_{2n-1} ,y_{2n})$

Since $h \in (0,1)$ it follows from (1.7) and (1.8) that $\{y_n\}$ is a Cauchy sequence. Since X is complete, $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the point z, that is as $n \to \infty$, we have Px_{2n}, ABx_{2n+1} and $STx_{2n+2} \to z$.

Since A and B are continuous and the pair {P, AB} is compatible of type (α) by proposition (1.3), we have as $n \to \infty$
$P(AB)x_{2n+1} = ABz$ and $(AB)^2x_{2n+1} = ABz$.

Now we take $x= ABx_{2n+1}$ and $y = x_{2n+2}$ in (1.6), we write
$\Phi(d(P(AB)x_{2n+1},Px_{2n+2}) , d((AB)^2x_{2n+1}, P(AB)x_{2n+1}), d((AB)^2x_{2n+1},STx_{2n+2}),$
$d(STx_{2n+2},Px_{2n+2}),d(STx_{2n+2},P(AB)x_{2n+1}),d((AB)^2x_{2n+1},Px_{2n+2})) \leq 0.$

Taking the limit $n \to \infty$, we have
$\Phi(d(ABz, z),d(ABz, ABz),d(ABz, z),d(z, z),d(z, ABz), d(ABz, z)) \leq 0.$

$\Phi(d(ABz,z),0,d(ABz,z),0,d(z,ABz),d(ABz,z)) \leq 0.$

which is a contradiction to (Φ_b). Thus ABz = z.

Now by (1.5), since $d(z,STz) \geq d(z,ABz) = 0$, we also have STz=z.

Therefore ABz = STz = z.
Again by putting $x = ABx_{2n+1}$ and y = z in (1.6), we write

$\Phi(d(P(AB)x_{2n+1},Pz), d((AB)^2x_{2n+1}, P(AB)x_{2n+1}), d((AB)^2x_{2n+1},STz),$
$d(STz, Pz), d(STz, P(AB)x_{2n+1}), d((AB)^2x_{2n+1}, Pz)) \leq 0.$

Taking the limit $n \to \infty$, we have
$\Phi(d(ABz, Pz),d(ABz, ABz),d(ABz, STz),d(STz, Pz), d(STz, ABz), d(ABz, Pz)) \leq 0.$

$\Phi(d(ABz,PZz),0,0,d(ABz,Pz),0,d(ABz,Pz)) \leq 0.$

which is contradiction to (Φ_c). Thus ABz = Pz.

Therefore ABz = STz = Pz =z.

Now we show that Bz = z. On putting x = Bz and y = z in (1.6), we write
$\Phi(d(P(Bz), Pz),d(AB(Bz), P(Bz)),d(AB(Bz), STz), d(STz, Pz),d(STz, P(Bz)), d(AB(Bz), Pz)) \leq 0,$

$\Phi(d(Bz,z),d(Bz,Bz),d(Bz,z),d(z,z),d(z,Bz),d(Bz,z) \leq 0,$

$\Phi(d(Bz,z),0,d(Bz,z),0,d(z,Bz),d(Bz,z) \leq 0,$

a contradiction to (Φ_b). Thus Bz = z. Hence Az =z.

Finally we show that Tz = z. By putting z = Tz and y = z in (1.6), we have

$\Phi(d(P(Tz), Pz),d(AB(Tz), P(Tz)),d(AB(Tz), STz), d(STz, Pz),d(STz, P(Tz)), d(AB(Tz), Pz)) \leq 0,$

$\Phi(d(Tz,z),d(Tz,Tz),d(Tz,z),d(z,z),d(z,Tz),d(Tz,z) \leq 0,$

$\Phi(d(Tz,z),0,d(Tz,z),0,d(z,Tz),d(Tz,z) \leq 0,$

a contradiction to (Φ_b). Thus Tz = z. Hence Sz =z.

Combining the above results, we get
Az = Bz = Sz = Tz = Pz =z.

Thus z is a common point of A, B, S, T and P.

For uniqueness let w ($z \neq w$) be another common fixed point of A, B, S, T and P. Then by (1.6), we write
$\Phi(d(Pz, Pw),d(ABz, Pz),d(ABz, STw),d(STw, Pw), d(STw, Pz), d(ABz, Pw)) \leq 0,$

$\Phi(d(z,w),d(z,z),d(z,w),d(w,w),d(w,z),d(z,w) \leq 0,$

$\Phi(d(z,w),0,d(z,w),0,d(w,z),d(z,w) \leq 0,$
a contradiction to (Φ_b). Thus z = w.
This completes the proof of the theorem.

Remark 1.1: In Theorem 1.1, if we replace the condition (1.5) by the following conditions
(1.9) A, B, S and T are continuous,

(1.10) the pairs {P, AB} and {P, ST}are compatible of type (α), then
Theorem 1.1 is still true.
By using Theorem 1.1, we have the following

Theorem 1.2: Let (X, d) be a complete metric space and A, B, S, T and $\{P_a\}_{a\in\Lambda}$ be Mappings from X into itself such that the conditions (1.3) and (1.4) hold and

(1.11) $\cup_{a\in\Lambda} P_a(X) \subset AB(X)$ and $\cup_{a\in\Lambda} P_a(X) \subset ST(X)$ where Λ is an index set,

(1.12) for all $a \in \Lambda$, $P_aB = BP_a$, $P_aT = TP_a$, $AB = BA$, $ST = TS$,

(1.13) for all $a \in \Lambda$, the pair $\{P_a, AB\}$ is compatible of type (α),

(1.14) the inequality

$\Phi(d(P_ax, P_ay), d(ABx, P_ax), d(ABx, STy), d(STy, P_ay),$
$d(STy, P_ax), d(ABx, P_ay)) \leq 0,$

for all $x,y \in X$, $a \in \Lambda$ and $\Phi \in \Psi$.

Then A, B, S, T and $\{P_a\}_{a\in\Lambda}$ have a unique common fixed point in X.

Remark 1.2: In Theorem 1.2, if we replace the condition (1.5) by the condition (1.9) and the following condition

(1.15) for all $a \in \Lambda$, the pair $\{P_a, AB\}$and $\{P_a, ST\}$is compatible of type (α), then Theorem 1.2 is still true.

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