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Expansive Mappings in Digital Metric Spaces

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Abstract

In this paper, we prove some common fixed point theorems for expansive mappings in digital metric spaces.

Keywords and phrases: Fixed point, digital image, expansive mappings, digital topology, digital metric spaces.
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1. INTRODUCTION

Topology is the study of geometric properties that does not depend upon shape of the objects, but rather how the points are connected to each other. In fact, topology deals with those properties of the objects that remain invariant under the continuous transformation of a given map. In 1979, Rosenfeld [14] introduced the concept of Digital Topology. Digital topology is concerned with geometrical and topological properties of digital image. Basically, digital topology involves the concept of adjacency (surrounding). Digital topology also provides a mathematical basis for image processing operations in 2D and 3D digital images. In recent times there have been many developments such as [1-13] in digital topology.

In topology, infinitely many points are considered in arbitrary small neighbourhood of a point, on the other hand, digital topology is concerned with finite number of points in a neighbourhood of a point. In fact, in digital topology neighbouring points are integers. Therefore, one can easily distinguish between general topology and digital topology by considering the neighbourhood of a point as shown in the folowing Figure 1.

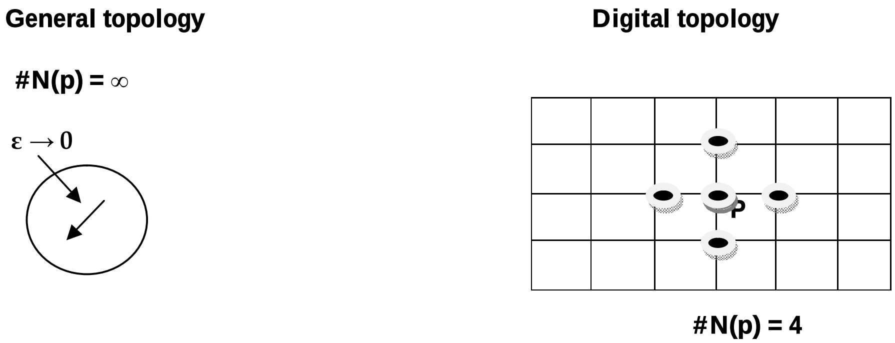


Figure 1: Neighboorhood in General and Digital topology

Digital image processing is a rapidly growing discipline in business, industry, medicine, environmental sciences and among many other fields. Digital image process involves the analysis of picture i.e., the regions of which it is composed of. A picture can be digitized into binary digits and one can obtain rectangular array of discrete values. The elements of these arrays are called pixels and the value of a pixel is called its gray level. The process of decomposing a picture into regions is called segmentation. Segmentation is basically a process of assigning the pixels. The one simple way of doing this process is called thresholding.

Once a picture has been segmented into regions then it can be described by properties of regions. Some of the properties of the regions depend on the gray levels of the points and some on the positions of the points.

2. TOPOLOGICAL STRUCTURE OF DIGITAL METRIC SPACES

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean n dimensional space with integer coordinates.

Definition 2.1. [4] Let l, n be positive integers with $1 \leq l \leq$ n. Consider two distinct points

$$p = (p_1, p_2, \dots, p_n),\ q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$$

The points p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j, $|p_j - q_j| \neq 1$, $p_j = q_j$.

(i) Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$ (see Figure 2).

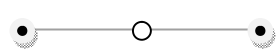


Figure 2: 2-adjacency

(ii) Two points p and q in $\mathbb{Z}^2$ are

(a) 8-adjacent if the points are distinct and differ by at most 1 in each coordinate i.e., the 4-neighbours of (x, y) are its four horizontal and vertical neighbours $(x \pm 1, y)$ and $(x, y \pm 1)$.

(b) 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate i.e., the 8- neighbours of (x, y) consist of its 4-neighbours together with its four diagonal neighbours $(x + 1, y \pm 1)$ and $(x - 1, y \pm 1)$. (see Figure 3).

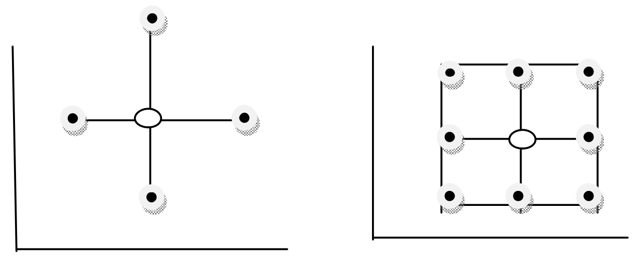
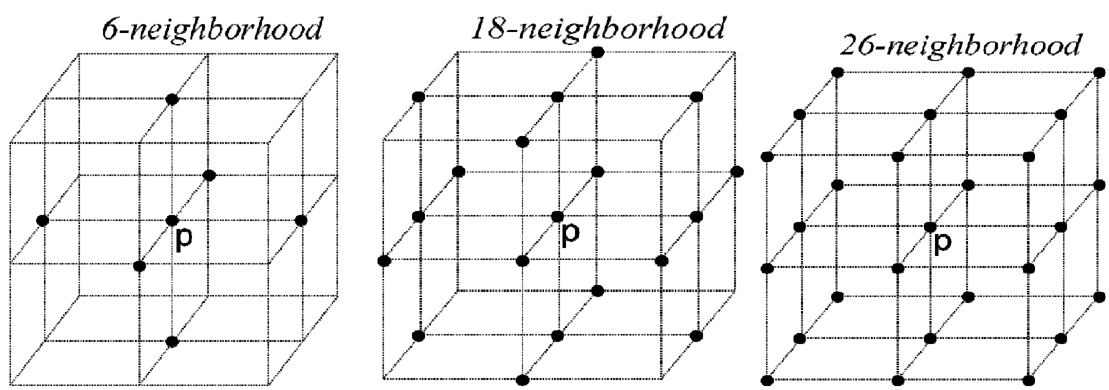


Figure 3: 4-adjacency and 8-adjacency

(iv) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate. i.e.,
(a) Six faces neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$
(b) Twelve edges neighbours $(x \pm 1, y \pm 1, z)$, $(x, y \pm 1, z \pm 1)$
(c) Eight corners neighbours $(x \pm 1, y \pm 1, z \pm 1)$
(v) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinate. i.e.,

(a)Twelve edges neighbours $(x \pm 1, y \pm 1, z)$, $(x, y \pm 1, z \pm 1)$
(b) Eight corners neighbours $(x \pm 1, y \pm 1, z \pm 1)$

(vi) Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if the points are 18-adjacent and differ in exactly one coordinate. i.e.,
(a) Six faces neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$ (See Figure 4).



6-adjacency 18-adjacency 26-adjacency

Figure 4: Adjacencies in $\mathbb{Z}^3$

Definition 2.2. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in \mathbb{N}$. A digital image is a pair (X, k), where k is an adjacency relation on X. Technically, a digital image (X, k) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0, x_1\} \subset X$ such that $x_0 \neq x_1$ and x_0 and x_1 are $k-$adjacent.

The notion of digital continuity in digital topology was developed by Rosenfeld [15] to study 2D and 3D digital images. Boxer [2] developed the digital version of several notions of topology and Ege and Karaca [7] studies Banach Contraction Principle in digital images.

Boxer [3] defined a k – neighbor of a point $p \in \mathbb{Z}^n$. It is a point of $\mathbb{Z}^n$ that is k - adjacent to p, where $k \in$ {2,4,6 8,18,26} and $n \in$ {1, 2, 3}.The set

$$N_k(p) = \{q \mid q \text{ is } k \text{ - adjacent to } p\}$$

is called the k -neighborhood of p. Boxer [2] defined a digital interval as

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\},$$

where a, b $\in \mathbb{Z}$ and a < b. A digital image $X \subset \mathbb{Z}^n$ is k-connected [11] if and only if for every pair of distinct points $x, y \in X$, there is a set $\{x_0, x_1, x_2, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ where x_i and x_{i+1} are k -neighbors and i = 0, 1... r- 1.

Definition 2.3. [3] Let $(X, k_0) \subset \mathbb{Z}^{n_0}$, $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images and f: $X \rightarrow Y$ be a function.
(i)If for every k_0-connected subset U of X, $f(U)$ is a k_1-connected subset of Y, then f is said to be (k_0, k_1)-continuous.
(ii) f is (k_0, k_1)-continuous for every k_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are k_1-adjacent in Y .
(iii) If f is (k_0, k_1)-continuous, bijective and f^{-1} is (k_0, k_1)-continuous, then f is called (k_0, k_1)-isomorphism and denoted by $\cong_{(k_0, k_1)} Y$.

Definition 2.4. Let (X, k) be a digital images set. Let d be a function from
$(X, k) \times (X, k) \rightarrow \mathbb{Z}^n$ satisfying the following:

(i) $d(x, y) \geq 0$;	(Non–negativity)
(ii) $d(x, y) = 0$ iff $x = y$;	(Identity)
(iii) $d(x, y) = d(y, x)$;	(Symmetry)
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.	(Triangle inequality)

The function d is called digital metric. The set (X, k) together with d is denoted by the triplet (X, d, k) called a digital metric space.

Proposition 2.5. [10] Let (X, d, k) be a digital metric space. A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) is
(i) a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that for all, $n, m \gneqq \alpha$, then

$$d(x_n, x_m) \lneqq 1 \text{ i.e., } x_n = x_m.$$

(ii) convergent to a point $l \in X$ if for all $\epsilon \gneqq 0$, there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$ then $d(x_n, l) \lneqq \epsilon, i.e. x_n = l$.

Proposition 2.6. [10] A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$, then $x_n = l$.

Theorem 2.7. [10] A digital metric space (X, d, k) is always complete.

Definition 2.8. [7] Let (X, d, k) be any digital metric space. A self map f on a digital metric space is said to be digital contraction, if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda\, d(x, y)$$

Proposition 2.9. [7] Every digital contraction map $f: (X, d, k) \rightarrow (X, d, k)$ is digitally continuous.

Proposition 2.10. [10] Let (X, d, k) be a digital metric space. Consider a sequence $\{x_n\} \subset X$ such that the points in $\{x_n\}$ are k adjacent. The usual distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most $\sqrt{t}$ depending on the position of the two points where $t \in Z^+$.

3. MAIN RESULT

In 1984, Wang et al. [16] introduced the notion of expansive mapping in metric spaces. The following definition in digital metric spaces is analogue to the notion of expansive mappings in metric spaces.

Definition 3.1 Suppose that (X, d, k) is a complete digital metric space and $S: X \to X$ be a mapping. The mapping S satisfy the condition $d(S(x), S(y)) \geq \alpha\, d(x, y)$ holds for all $x, y \in X$ and $\alpha > 1$, then S is called a digital expansive mapping.

Theorem 3.2 Let, (X, d, k) be a digital metric space and S be an onto continuous self map on X satisfy the following

$$d\big(S(x), S(y)\big) \geq \alpha\mu$$

where $\alpha > 1$, and $\mu \equiv \mu(x, y) = max\left\{d(x, y), \frac{d(x,S(x))+d(y,S(y))}{2}, \frac{\{d(x,S(y))+d(y,S(x))\}}{2}\right\}$ then S has a fixed point.

Proof: Let $x_0 \in X$. Since S is onto, there exists an element x_1 satisfying $x_1 \in S^{-1}(x_0)$. By induction, one can choose, $x_n \in S^{-1}(x_{n-1})$ where $(n = 2,3,4, \dots)$.

If $x_{m-1} = x_m$ for some m, then x_m is a fixed point of S. Without loss of generality, we can suppose $x_n \neq x_{n-1}$ for every n. So,

$$d(x_{n-1}, x_n) = d\big(S(x_n), S(x_{n+1})\big) \geq \alpha\mu_n$$

where, $\mu_n = max\left\{d(x_n, x_{n+1}), \frac{d(x_n,S(x_n))+d(x_{n+1},S(x_{n+1}))}{2}, \frac{d(x_n,S(x_{n+1}))+d(x_{n+1}S(x_n))}{2}\right\}$

$$= \left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}, \frac{d(x_{n+1}, x_{n-1})}{2}\right\}$$

Case1: $d(x_{n-1}, x_n) \geq \alpha d(x_n, x_{n+1})$

Case2: $d(x_{n-1}, x_n) \geq \alpha \frac{d(x_n,x_{n-1})+d(x_n,x_{n+1})}{2}$

$d(x_{n-1}, x_n) - \left(\frac{\alpha}{2} d(x_{n-1}, x_n)\right) \geq \frac{\alpha}{2} d(x_n, x_{n+1}),$

$\left(\frac{2-\alpha}{\alpha}\right) d(x_{n-1}, x_n) \geq d(x_n, x_{n+1}),$

$d(x_{n-1}, x_n) \geq kd(x_n, x_{n+1})$ where $k = \frac{\alpha}{2-\alpha} > 1$ i.e., $d(x_n, x_{n+1}) \leq \frac{1}{k} d(x_{n-1}, x_n)$(since, if $1 < \alpha$, then

clearly $k > 1$ and if $\alpha \geq 2$ then we get $d(x_n, x_{n+1}) < 0$ which is a contradiction)

Case3: $d(x_{n-1}, x_n) \geq \alpha \frac{d(x_{n+1},x_{n-1})}{2}$
which is same as Case2, since $d(x_{n+1}, x_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$

So, in each case $d(x_{n-1}, x_n) \geq \emptyset d(x_n, x_{n+1})$ for some $\emptyset > 1$.

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \dots + d(x_{n+m-1}, x_{n+m}) \leq \left(\frac{1}{\emptyset}\right)^{n+1} d(x_0, x_1) + \dots + \left(\frac{1}{\emptyset}\right)^{n+m} d(x_0, x_1)$$

$$\leq \frac{\left(\frac{1}{\emptyset}\right)^{n+1}}{1 - \frac{1}{\emptyset}} d(x_0, x_1)$$

Clearly, $\{x_n\}$ is a Cauchy sequence in X. Since, X is complete $\{x_n\}$ converges to some $x \in X$. Since, S is onto there exists $y \in X$ such that $y \in S^{-1}(x)$ and for infinitely many n, $x_n \neq x$, for such n

$$d(x_n, x) = d\big(S(x_{n+1}), S(y)\big) \geq \alpha\mu(x_{n+1}, y)$$

Now, $\mu(x_{n+1}, y) = max\left\{d(x_{n+1}, y), \frac{d(x_{n+1}, S(x_{n+1})) + d(y, S(y))}{2}, \frac{d(x_{n+1}, S(y)) + d(y, S(x_{n+1}))}{2}\right\}$

On taking limit as $n \to \infty$, we get,

$$\mu(x, y) = max\left\{d(x, y), \frac{d\big(x, S(x)\big) + d\big(y, S(y)\big)}{2}, \frac{d\big(x, S(y)\big) + d\big(y, S(x)\big)}{2}\right\}$$

$$= \left\{d(x, y), \frac{d\big(x, S(x)\big) + d(x, y)}{2}, \frac{d\big(y, S(x)\big)}{2}\right\}$$

Also, $0 \geq k\mu(x, y)$, which implies $\mu(x, y) = 0$. So, $d(x, y) = 0$ or $d\big(y, S(x)\big) = 0$.

If $d(x, y) = 0$, then $x = y$.

If $d\big(y, S(x)\big) = 0$, then $y = S(x)$ but $x = S(y)$ and S is continuous. Therefore, $x = y$.

(Since, by continuity of S, we get $y = \lim_{n\to\infty} S(x_n)$. Therefore, $S(x_n)$ converges to y. Then, there exist $N \in \mathbb{N}$, such that, $\forall\, n > N, S(x_n) \in S^{-1}(x)$, by construction $x_{n+1} \in S^{-1}(x)$ and hence by uniqueness of limit $x = y$)

Thus, T has a fixed point in X.

Theorem 3.3 Let, (X, d, k) be a complete digital metric space and S be an onto self map which is continuous on X. Let S satisfy the condition

$$d\big(S(x), S(y)\big) \geq \alpha\mu$$

where $\alpha > 1$, and $\mu \equiv \mu(x, y) = max\left\{d(x, y), \frac{d(x, S(x)) + d(y, S(y))}{2}, d\big(x, S(y)\big), d\big(y, S(x)\big)\right\}$ then S has a fixed point.

Proof: Let $x_0 \in X$, since S is onto, there exists an element x_1 satisfying $x_1 \in S^{-1}(x_0)$. By the same way, we can choose, $x_n \in S^{-1}(x_{n-1})$ where $(n = 2,3,4, \ldots)$.

If $x_{m-1} = x_m$ for some m, then x_m is a fixed point of S. Without loss of generality, we can suppose $x_n \neq x_{n-1}$ for every n. So,

$$d(x_{n-1}, x_n) = d\big(S(x_n), S(x_{n+1})\big) \geq \alpha\mu_n$$

where, $\mu_n = max\left\{d(x_n, x_{n+1}), \frac{d(x_n, S(x_n)) + d(x_{n+1}, S(x_{n+1}))}{2}, d\big(x_n, S(x_{n+1})\big), d\big(x_{n+1}, S(x_n)\big)\right\}$

$$= \left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}, d(x_{n+1}, x_{n-1})\right\}$$

Case1: $d(x_{n-1}, x_n) \geq \alpha d(x_n, x_{n+1})$

Case2: $d(x_{n-1}, x_n) \geq \alpha \frac{d(x_n, x_{n-1}) + d(x_n, x_{n+1})}{2}$

$d(x_{n-1}, x_n) - \left(\frac{\alpha}{2} d(x_{n-1}, x_n)\right) \geq \frac{\alpha}{2} d(x_n, x_{n+1})$,

$\left(\frac{2-\alpha}{\alpha}\right) d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$,

$d(x_{n-1}, x_n) \geq k d(x_n, x_{n+1})$ where $k = \frac{\alpha}{2-\alpha} > 1$

(since, if $1 < \alpha < 2$, then clearly $k > 1$ and if $\alpha \geq 2$ then we get $d(x_n, x_{n+1}) < 0$ which is a contradiction)

Case3: $d(x_{n-1}, x_n) \geq \alpha d(x_{n+1}, x_{n-1})$

which is same as Case2, since $d(x_{n+1}, x_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$

So, in each case $d(x_{n-1}, x_n) \geq \emptyset d(x_n, x_{n+1})$ for some $\emptyset > 1$.

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \leq \left(\frac{1}{\emptyset}\right)^{n+1} d(x_0, x_1) + \cdots + \left(\frac{1}{\emptyset}\right)^{n+m} d(x_0, x_1)$$
$$\leq \frac{\left(\frac{1}{\emptyset}\right)^{n+1}}{1 - \frac{1}{\emptyset}} d(x_0, x_1)$$

Clearly, $\{x_n\}$ is a Cauchy sequence in X. Since, X is complete $\{x_n\}$ converges to some $x \in X$. Since S is onto there exists $y \in X$ such that $y \in S^{-1}(x)$ and for infinitely many n, $x_n \neq x$, for such n

$$d(x_n, x) = d\big(S(x_{n+1}), S(y)\big) \geq \alpha\mu(x_{n+1}, y)$$

Now, $\mu(x_{n+1}, y) = max\left\{d(x_{n+1}, y), \frac{d(x_{n+1}, S(x_{n+1})) + d(y, S(y))}{2}, d\big(x_{n+1}, S(y)\big), d\big(y, S(x_{n+1})\big)\right\}$

On taking limit as $n \to \infty$, we get,

$$\mu(x, y) = max\left\{d(x, y), \frac{d\big(x, S(x)\big) + d\big(y, S(y)\big)}{2}, d(x, x), d\big(y, S(x)\big)\right\}$$
$$= \left\{d(x, y), \frac{d\big(x, S(x)\big) + d(x, y)}{2}, d\big(y, S(x)\big)\right\}$$

Also, $0 \geq k\mu(x, y)$, which implies $\mu(x, y) = 0$. So, $d(x, y) = 0$ or $d\big(y, S(x)\big) = 0$.

If $d(x, y) = 0$, then $x = y$.

If $d\big(y, S(x)\big) = 0$, then $y = S(x)$ but $x = S(y)$ and S is continuous. So, $x = y$.

Thus, S has a fixed point in X.

Corollary 3.4: If the map T in the theorems (3.2) and (3.3) be bijective, then T has a unique fixed point.

4. SCOPE OF THE STUDY

The aim of this paper is to discuss the digital version of some fixed point theorems concerning expansive maps. Somehow the results in this paper are useful in zooming out of digital images and the array of digital images. In future, some other properties of digital images can be discussed with the viewpoint of fixed point theory.

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