

INDEPENDENT VERIFICATION OF EULER'S FORMULA IN GRAPH THEORY FOR N IDENTICAL CIRCLES

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Abstract

This paper discusses the famous Euler's formula in graph theory and its independent verification for the case of n identical circles forming a planar graph. A planar connected graph is generated using n circles, and an intuitive approach is taken to construct formulae for the number of vertices (v), edges (e) and faces (f), where symmetry of the pattern is used to determine the relation between v , e and f .

Keywords: vertices, edges, faces, Planar graph, disconnected graph, intuitive analysis.

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1. INTRODUCTION

Leonhard Euler (b. April 15, 1707- d. September 18, 1783), a Swiss mathematician, physicist, astronomer, logician and engineer is well known for his influential discoveries in the field on mathematics and physics. In 1736, he presented a formal solution to a problem, known as the Seven Bridges of Königsberg. The problem is to decide whether it is possible to follow a path that crosses each of the bridges exactly once and return to the starting point. A negative resolution to this was proved by Euler, which is historically considered to be the first theorem of 'Graph Theory'. As he laid the foundations of graph theory, Euler came up with a formula relating the number of vertices (V), number of edges (E) and number of faces (F) for a planer graph, given as:

$$V - E + F = 2 \quad \dots\dots\dots (1)$$

The constant '2' appearing in this formula on the R. H. S. is now known as the 'Euler characteristic' for a convex polyhedron

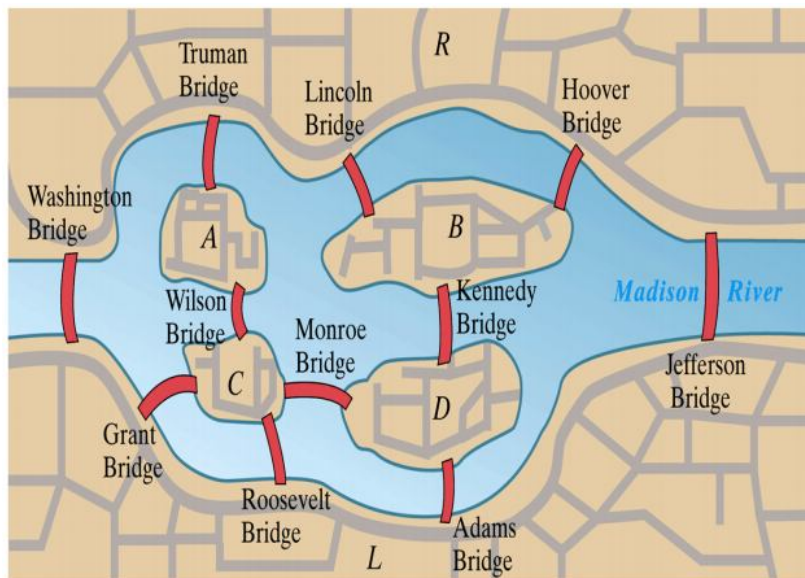


Figure (1): The seven bridges of Königsberg

2. PLANAR GRAPH

A graph is called planar, if it can be drawn in a plane without crossing the edges. To put in formal words, a graph is planar if it has an embedding in the plane, where each vertex is mapped to a distinct point $P(v)$, and edge (u,v) to simple curves connecting $P(u)$ and $P(v)$ such that the curves intersect only at their endpoints i. e. the vertices. Along with vertices and edges, there is another important notion for a planar graph: a 'face'. It is a region of the plane bounded by two edges of the graph.

3. THE THEOREM: FORMAL STATEMENT WITH ILLUSTRATIONS

Statement: If a finite, connected, planar graph is drawn in the plane without any intersections, and v is the number of vertices, e is the number of edges and f is the number of faces, then

$$v + f = e + 2 \quad \dots\dots\dots (2)$$

Let us go through an illustration to understand equation (2). Let us consider the K_4 graph described in figure (2).

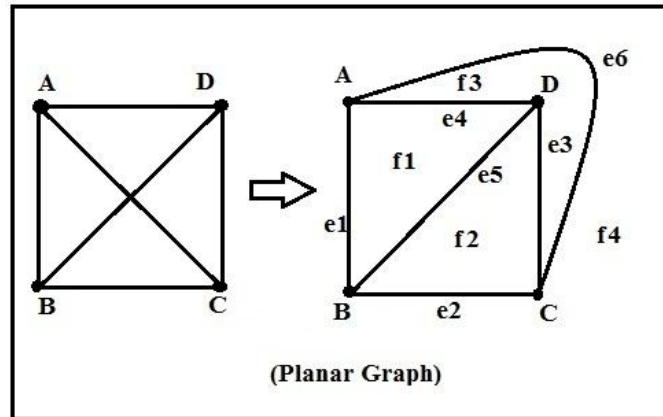


Figure (2): K_4 planar graph

At first, edges AC and BD appear to intersect, but one can find an alternate way to connect the vertices A and C such that the edge AC does not intersect edge BD, and the graph is still a planar graph.

Here, Vertex set $V = \{A, B, C, D\}$ and Edge set is $E = \{AB, AD, AC, BD, BC, DC\}$. The faces formed are marked by index f_i , $i = 1, 2, 3, 4$.

Thus, $v = 4$, $e = 6$, $f = 4$. This satisfies the equation (2), as $4 + 4 = 6 + 2$.
Now, consider the following disconnected graph:

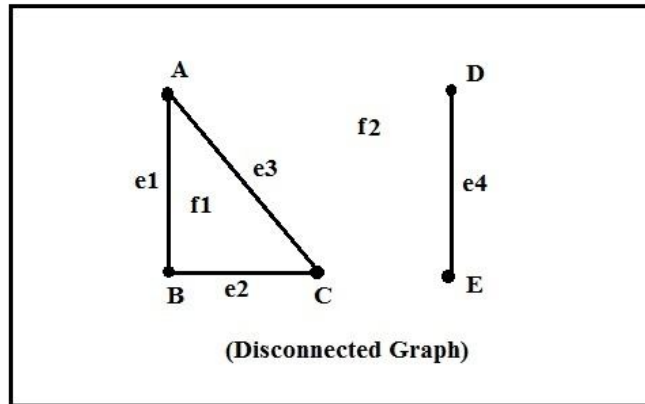


Figure (3): A disconnected graph

Here, Vertex set is $V = \{A, B, C, D, E\}$ and Edge set is $E = \{AB, AC, BC, DE\}$. The faces formed are marked by indices f_1 and f_2 .

Thus, $v = 5$, $e = 4$, $f = 2$, which does not satisfy the equation (2), as $5 + 2 \neq 4 + 2$.

4. THE CASE OF 'N' IDENTICAL CIRCLES

We propose to verify the Euler's formula for a planar connected graph constructed using intersection of n identical circles such that each circle intersects every other circle only in two distinct points to form a vertex at each intersection.

We begin the discussion by initializing the number of circles n to 2, the minimum value n must have to form an intersection. The situation is shown in figure (4).

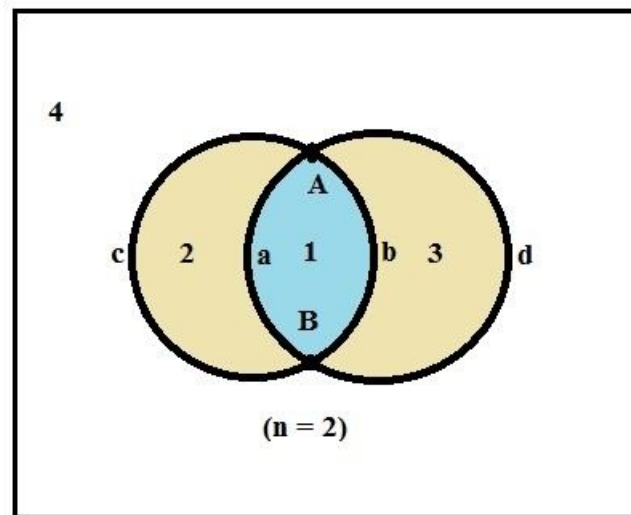


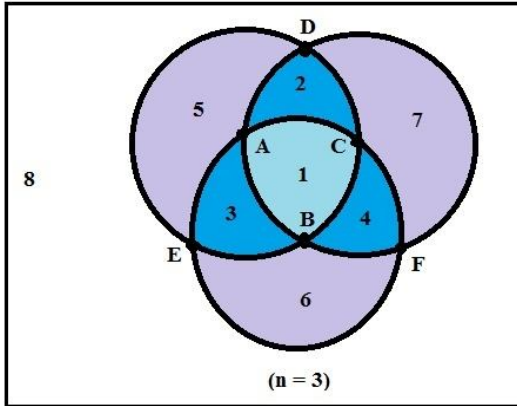
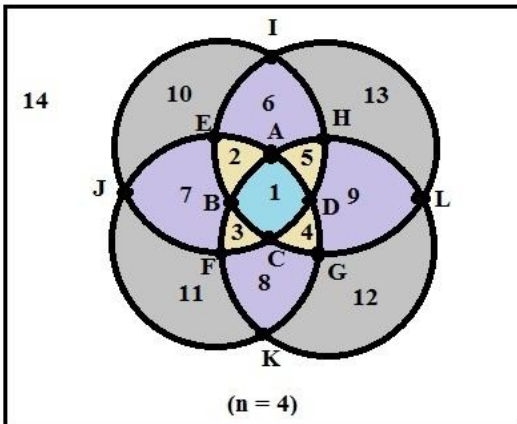
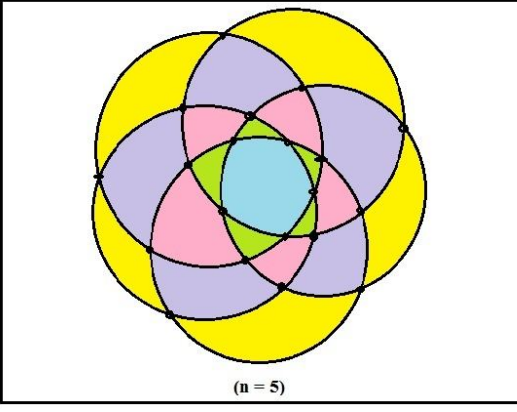
Figure (4): Two identical circles intersecting at two distinct points

Here, the vertex set is $V = \{A, B\} \rightarrow v = 2$. The edge set is $E = \{AcB, AbB, AaB, AdB\} \rightarrow e = 4$. The faces are marked by $F = \{1, 2, 3, 4\} \rightarrow f = 4$. Clearly, the formula holds true.

The counting becomes critical when the number of circles increases. So, we adopt a method to count the parameters viz. vertices, edges and faces as follows: Initially, one has to count the relevant quantities corresponding to the region which is common to all the circles (say, central region). Then, the same is to be counted for the petals originating from the edges of the central region. This process has to be repeated till all the vertices and edges are exhausted.

Table (1): generalizing the discussion for n circles

No. of circles	Description	Counting
2		$v = 1 + 1 = 2 * 1 = 2$ $e = 1 * 2 + 1 * 2 = 4$ $f = 1 + 2 + 1 = 4$ $\therefore v + f = e + 2$ $\therefore \text{Euler's formula holds true}$

3	 <p>(n = 3)</p>	$v = 3 + 3 = 3 * 2 = 6$ $e = 1 * 3 + 2 * 3 + 1 * 3 = 12$ $f = 1 + 3 + 3 + 1 = 8$ $\therefore v + f = e + 2$ <p>∴ Euler's formula holds true</p>
4	 <p>(n = 4)</p>	$v = 4 + 4 + 4 = 4 * 3 = 12$ $e = 1 * 4 + 2 * 4 + 2 * 4 + 1 * 4 = 24$ $f = 1 + 4 + 4 + 4 + 1 = 14$ $\therefore v + f = e + 2$ <p>∴ Euler's formula holds true</p>
5	 <p>(n = 5)</p>	$v = 5 + 5 + 5 + 5$ $= 5 * 4$ $= 5 * (5 - 1)$ $= 20$ $e = 1 * 5 + 2 * 5 + 2 * 5 + 2 * 5 + 1 * 5$ $= 40$ $f = 1 + 5 + 5 + 5 + 5 + 5 + 1$ $= 22$ $\therefore f + v = e + 2$ <p>∴ Euler's formula hold true</p>
.	.	.
n	<p>Generalizing the above discussion, for n circles, we write:</p> $v = n + n + n + \dots (n - 1) \text{ times} = n (n - 1) \dots\dots\dots (3)$ $e = n + \{ 2 * n + 2 * n + 2 * n + \dots (n - 2) \text{ times} \} + n = 2n + 2n (n - 2) \dots\dots (4)$ $f = 1 + \{ n + n + n + \dots (n - 1) \text{ times} \} + 1 = 2 + n (n - 1) \dots\dots\dots (5)$	

$f + v = 2 + n(n - 1) + n(n - 1) = 2 + (n + n)(n - 1) = 2 + 2n(n - 1)$ <p>And,</p> $e + 2 = 2n + 2n(n - 2) + 2 = 2 + 2n(1 + n - 2) = 2 + 2n(n - 1)$ <p>From (6) and (7), $f + v = e + 2$, and thus, Euler's formula holds true.</p>	<p>..... (6)</p> <p>..... (7)</p>
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A similar intuitive analysis for n identical squares and n identical equilateral triangles as well, in order to verify the Euler's formula, yields the same results as given by the set of equations (3) to (7).

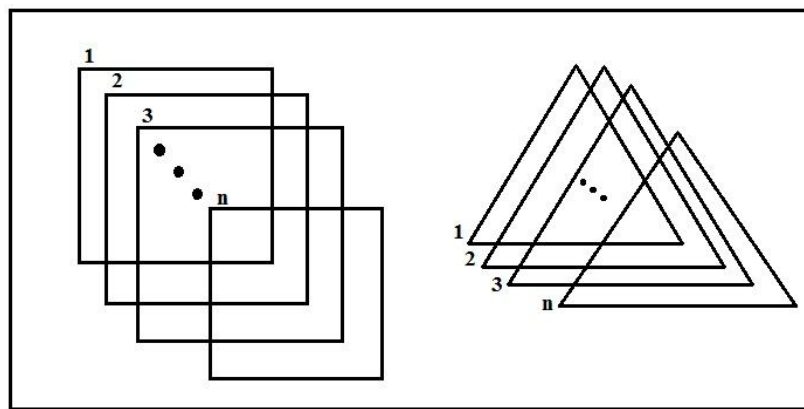


Figure (5): The case of n identical squares and equilaterals with exactly two intersections

5. ACKNOWLEDGEMENTS

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6. REFERENCES

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