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ON THE DEGENERATE LAPLACE TRANSFORM - I

Lalit Mohan Upadhyaya*

Author Affiliation:
Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, (Uttarakhand), India-248179.

E-mail: lmupadhyaya@rediffmail.com; hetchres@gmail.com

***Corresponding Author:**
Lalit Mohan Upadhyaya, Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, (Uttarakhand), India-248179.

E-mail: lmupadhyaya@rediffmail.com; hetchres@gmail.com

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Abstract

We make an attempt in this paper to extend the very recent work of Kim and Kim [24] on degenerate Laplace transform. Using the definition of the degenerate Laplace transform as given by Kim and Kim [24] and some of the results developed by them in this study we prove the first and second translation theorems and the change of scale property for the degenerate Laplace transform and give some illustrations highlighting the applications of these results for the degenerate Laplace transform of a function.

Keywords: Degenerate Laplace transform, first translation, second translation, change of scale property.

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1. INTRODUCTION

The courses in operational calculus form an essential part of the curricula of mathematics worldwide in universities and engineering colleges because of the wide spread applications of these concepts in solving a number of problems arising in engineering, physics, applied mathematics and other related disciplines of scientific study. The concept of Laplace transform is invariably taught as an integral part of this course of operational calculus to the mathematicians, engineers, physicists and scientists from related disciplines on account of the powerful applications of this mathematical tool in engineering, physics and applied mathematics problems. The impact of Laplace transforms in the study of mathematics has been very huge and they form an integral part of a number of monumental works in mathematics, some of the classical ones are [4,5,6,25], they also form parts of many standard and authoritative mathematical handbooks like Gradshteyn, Ryzhik [15], Schaum's handbook [8] besides many others.

We follow the notations used by Kim and Kim [24] for denoting the Laplace transform and the degenerate Laplace transform of a function throughout this paper. The Laplace transform of a function $f(t)$ of the variable t defined for $t > 0$, denoted by $\mathbf{L}\{f(t)\}$, is defined by the integral (see, for instance, (1), p.1 [7] or, (32.1), p. 161 [8], or, (3.3), p.103 [26])

$$\mathbf{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \tag{1.1}$$

provided the integral in (1.1) converges for some value of the complex parameter s. For sufficient conditions for the existence of the Laplace transform of a function $f(t)$, the reader is referred to the Theorem 1-1, p.2 and the Problem 145, p.38 of [7] or, Theorem 3.1, p. 103 [26].

The above concept of the Laplace transform of a function has been generalized in many ways. Chung, Kim and Kwon [16] have recently given the $q-$ analog of the Laplace transform. The recent developments in the theory of special functions of matrix arguments with real symmetric positive definite matrices and Hermitian positive definite matrices as arguments through the Laplace transform technique may be seen from the work of Mathai (Chapters 5 and 6) [14]. Some integrals involving the Laplace transforms of Appell's and Humbert's functions of matrix arguments are available in the works of this author [9-13]. Carlitz [2] studied the degenerate Sterling, Bernoulli and Eulerian numbers besides giving a degenerate analogue of the Staudt-Clausen theorem [1]. Most recently, the degenerate special polynomials and numbers have been very extensively studied by T. Kim and his research collaborators D.V. Dolgy, J.J. Seo, L.C. Jang, H.-I. Kwon, and D.S. Kim [17-24].

In this paper we aim to extend the very recent study of Kim and Kim [24] and using their various definitions and results of this paper [24] we state and prove the first and second translation theorems and the change of scale property for the degenerate Laplace transform of a given function. The paper is divided in two sections: besides giving an introduction of the necessary concepts in the first section of the paper we also state the preliminary results in this section which shall be invoked by us in proving our main results in the second section of the paper.

Definition 1.1: The Degenerate Exponential Function (Kim and Kim [24], (1.3), p. 241) – The degenerate exponential function, represented by e_λ^t, is a function of two variables λ and t, where, $\lambda \in (0, \infty)$, $t \in \mathbb{R}$ and is defined by

$$e_\lambda^t = (1 + \lambda t)^{\frac{1}{\lambda}} \tag{1.2}$$

It is important to note here that this definition generalizes the classical exponential function e^t defined by the well known series relation

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{1.3}$$

because one can easily deduce from (1.2) that (see Kim and Kim [24], p.241)

$$\lim_{\lambda \to 0+} e_\lambda^t = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \tag{1.4}$$

The well known Euler's exponential formula is given by (see, for instance, (1.7) p.241 Kim and Kim [24])

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5}$$

where $i = \sqrt{-1}$. From this follow immediately the definitions of the elementary trigonometric functions sine and cosine in terms of the exponential function as (see, for instance, (1.8) p.241 Kim and Kim[24])

$$\cos a\theta = \frac{e^{ia\theta} + e^{-ia\theta}}{2}, \quad \sin a\theta = \frac{e^{ia\theta} - e^{-ia\theta}}{2i} \tag{1.6}$$

Definition 1.2: The Degenerate Euler Formula: The degenerate Euler formula is defined by the relation (see (1.9), p.242 Kim and Kim[24])

$$e_\lambda^{it} = (1+\lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i\sin_\lambda(t) \tag{1.7}$$

From (1.7) it can be readily inferred that (see (1.10) p.242 Kim and Kim [24])

$$\lim_{\lambda\to 0+} e_\lambda^{it} = \lim_{\lambda\to 0+} (1+\lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos t + i\sin t \tag{1.8}$$

Further it is rather easy to note from (1.7) and (1.8) that (see (1.11) p.242 Kim and Kim[24])

$$\lim_{\lambda\to 0+} \cos_\lambda(t) = \cos t, \lim_{\lambda\to 0+} \sin_\lambda(t) = \sin t \tag{1.9}$$

Definition 1.3: The Degenerate Cosine and Degenerate Sine Functions: The following respective definitions of the degenerate cosine and degenerate sine functions follow from (1.7) (see (1.12), p.242 Kim and Kim[24])

$$\cos_\lambda(t) = \frac{e_\lambda^{it} + e_\lambda^{-it}}{2}, \sin_\lambda(t) = \frac{e_\lambda^{it} - e_\lambda^{-it}}{2i} \tag{1.10}$$

Definition 1.4: The Degenerate Laplace Transform of a Function: (Kim and Kim [24], (3.1), p.244) Let $f(t)$ be a function defined for $t \geq 0$ and let $\lambda \in (0,\infty)$, then the degenerate Laplace transform of the function $f(t)$, represented by $F_\lambda(s)$, is defined by the integral

$$\mathbf{L}_\lambda\{f(t)\} = F_\lambda(s) = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)\,dt \tag{1.11}$$

Definition 1.5: The Degenerate Gamma Function: (Kim and Kim [24], (2.1), p.242)- The degenerate gamma function, denoted by $\Gamma_\lambda(s)$, for the complex variable s with $0 < \operatorname{Re}(s) < \frac{1}{\lambda}$ is defined by the integral (provided the integral converges)

$$\Gamma_\lambda(s) = \int_0^\infty (1+\lambda t)^{\frac{-1}{\lambda}} t^{s-1}\,dt = \int_0^\infty e_\lambda^{-t} t^{s-1}\,dt \tag{1.12}$$

It is not very difficult to infer from (1.12) that, $\Gamma_\lambda(s)$ provides a generalization to the well known gamma function $\Gamma(s)$ defined for a complex variable s by (see, for instance, (1), section (1.1), Chapter 1, p.1 [4])

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt, \quad \operatorname{Re}(s) > 0 \tag{1.13}$$

where $\operatorname{Re}(s)$ denotes the real part of the complex variable s. From (1.13) we readily observe that

$$\Gamma(s+1) = s\Gamma(s) \quad \text{and} \quad \Gamma(n+1) = n!, \quad n \in \mathbb{N} \tag{1.14}$$

(see, for example, (1), (4) section (1.2), Chapter 1, p.3 [4]). We mention that in the limiting case $\lambda \to 0+$, $\Gamma_\lambda(s)$ of (1.12) reduces to $\Gamma(s)$ of (1.13).

Kim and Kim [24] have shown that (see (2.2), p.242 of [24])

$$\Gamma_\lambda(s) = \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right) \tag{1.15}$$

where $B(x,y)$ is the well known beta function given by (see, for example, (2), (5) section 1.5, Chapter 1 [4])

$$B(x,y) = \int_0^\infty v^{x-1}(1+v)^{-x-y}\,dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \operatorname{Re}(x), \operatorname{Re}(y) > 0 \tag{1.16}$$

Another important result for the degenerate gamma function proved by Kim and Kim (see Theorem 2.1, p.243 [24]) is as mentioned below which holds for $\lambda \in (0,1)$ and $0 < \operatorname{Re}(s) < \frac{1-\lambda}{\lambda}$

$$\Gamma_{\lambda}(s+1)=\frac{s}{(1-\lambda)^{s+1}}\Gamma_{\frac{\lambda}{1-\lambda}}(s) \quad (1.17)$$

which is clearly a generalization of (1.14) to which it reduces when $\lambda \to 0+$.

2. RESULTS FOR THE DEGENERATE LAPLACE TRANSFORM OF A FUNCTION

We now proceed to discuss three results for the degenerate Laplace transform of a function $f(t)$ as outlined in the abstract of this paper. These results will be stated below as theorems and their proofs will be provided by utilizing the concepts and definitions as stated in the first section of the paper.

Theorem 2.1: The First Translation Theorem: If $\mathbf{L}_{\lambda}\{f(t)\}=F_{\lambda}(s)$, then

$$\mathbf{L}_{\lambda}\{e_{\lambda}^{at} f(t)\}=F_{\lambda}(s-a) \quad (2.1)$$

Proof: The proof immediately follows by applying the definition 1.4 to the left side of (2.1), keeping in mind the definition (1.1) as follows

$$\mathbf{L}_{\lambda}\{e_{\lambda}^{at} f(t)\}=\int_{0}^{\infty} e_{\lambda}^{-st} e_{\lambda}^{at} f(t)\,dt=\int_{0}^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}}(1+\lambda t)^{\frac{a}{\lambda}} f(t)\,dt$$

$$=\int_{0}^{\infty}(1+\lambda t)^{\frac{-(s-a)}{\lambda}} f(t)\,dt=F_{\lambda}(s-a)$$

This theorem generalizes the Theorem 1-3 Chapter 1, p.3 of [7] or the Theorem 3.3, p.107 of [26] . It can be seen that in the limiting case as $\lambda \to 0+$ in (2.1) then it reduces to the Theorem 1-3 Chapter 1, p.3 of [7] or the Theorem 3.3, p.107 of [26].

Kim and Kim [24] have shown that (see (3.8), p.244)

$$\mathbf{L}_{\lambda}\{\cos_{\lambda}(at)\}=\frac{s-\lambda}{(s-\lambda)^{2}+a^{2}} \quad (2.2)$$

As an illustration of the first translation theorem, we can write from (2.1) and (2.2) that, if $f(t)=\cos_{\lambda}(bt)$ in (2.1), then

$$\mathbf{L}_{\lambda}\{e_{\lambda}^{at}\cos_{\lambda}(bt)\}=\frac{s-\lambda-a}{(s-\lambda-a)^{2}+b^{2}} \quad (2.3)$$

Similarly, Kim and Kim [24] have proved that (see (3.9), p.245)

$$\mathbf{L}_{\lambda}\{\sin_{\lambda}(at)\}=\frac{a}{(s-\lambda)^{2}+a^{2}} \quad (2.4)$$

from where follows immediately, with the aid of (2.1) that

$$\mathbf{L}_{\lambda}\{e_{\lambda}^{at}\sin_{\lambda}(bt)\}=\frac{b}{(s-\lambda-a)^{2}+b^{2}} \quad (2.5)$$

Theorem 2.2: The Second Translation Theorem: If $\mathbf{L}_{\lambda}\{f(t)\}=F_{\lambda}(s)$, and

$$g(t)=\begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$$

then

$$\mathbf{L}_{\lambda}\left\{g(t)\right\} = e_{\lambda}^{-sa}\mathbf{L}_{\frac{\lambda}{1+\lambda a}}\left\{\left(1+\frac{\lambda t}{1+\lambda a}\right)^{sa} f(t)\right\} \tag{2.6}$$

Proof: We see that

$$\mathbf{L}_{\lambda}\left\{g(t)\right\} = \int_0^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}} g(t)dt = \int_0^{a}(1+\lambda t)^{\frac{-s}{\lambda}} g(t)dt + \int_a^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}} g(t)dt$$
$$= \int_0^{a}(1+\lambda t)^{\frac{-s}{\lambda}}.0.dt + \int_a^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}} f(t-a)dt$$

If we put $t-a=u$ (so that, $dt=du$) in the second integral on the right side of the above equation, then, we get

$$\mathbf{L}_{\lambda}\left\{g(t)\right\} = \int_0^{\infty}\left(1+\lambda(u+a)\right)^{\frac{-s}{\lambda}} f(u)du = (1+\lambda a)^{\frac{-s}{\lambda}}\int_0^{\infty}\left(1+\frac{\lambda u}{1+\lambda a}\right)^{\frac{-s}{\lambda}} f(u)du$$

On taking $\mu = \dfrac{\lambda}{1+\lambda a}$ and keeping in mind (1.2), the above relation can be rewritten as

$$\mathbf{L}_{\lambda}\left\{g(t)\right\} = e_{\lambda}^{-sa}\int_0^{\infty}(1+\mu u)^{\frac{-s(1-\mu a)}{\mu}} f(u)du = e_{\lambda}^{-sa}\int_0^{\infty}(1+\mu u)^{\frac{-s}{\mu}}(1+\mu u)^{sa} f(u)du$$

Or,

$$\mathbf{L}_{\lambda}\left\{g(t)\right\} = e_{\lambda}^{-sa}\mathbf{L}_{\mu}\left\{(1+\mu u)^{sa} f(u)\right\} \tag{2.7}$$

On changing the dummy variable u by t and substituting the value of μ on the right side of (2.7) we arrive at the right side of (2.6).

This theorem generalizes the Theorem 1-4, p.3 Chapter 1 of [7] or the Theorem 3.4, p.107 of [26]. Obviously as $\lambda \to 0+$ in (2.6) then it reduces to the Theorem 1-4 Chapter 1, p.3 of [7] or the Theorem 3.4, p.107 of [26].

Kim and Kim [24] have proved the following result (see (3.19), p. 246 Kim and Kim [24]) for $\alpha \in \square$ with $\alpha > -1$ and for $s > (\alpha+1)\lambda$

$$\mathbf{L}_{\lambda}\left\{t^{\alpha}\right\} = \frac{1}{s^{\alpha+1}}\Gamma_{\frac{\lambda}{s}}(\alpha+1) \tag{2.8}$$

which in the limiting case $\lambda \to 0+$ reduces to the long known elementary result in the theory of Laplace transforms,

$$\mathbf{L}\left\{t^{\alpha}\right\} = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1) = \frac{\alpha!}{s^{\alpha+1}}, \qquad \text{when} \quad \alpha \in \square \tag{2.9}$$

As an illustration for the Theorem 2.2, let us take $f(t) = t^{\alpha}$ for $\alpha > -1$ and $s > (\alpha+1)\lambda$, and define

$$g(t) = \begin{cases} (t-a)^{\alpha}, & t > a \\ 0, & t < a \end{cases}$$

Then from the Theorem 2.2 follows

$$\mathbf{L}_\lambda\left\{g(t)\right\}=e_\lambda^{-sa}\mathbf{L}_{\frac{\lambda}{1+\lambda a}}\left\{\left(1+\frac{\lambda t}{1+\lambda a}\right)^{sa}t^\alpha\right\}$$

$$=e_\lambda^{-sa}\mathbf{L}_\mu\left\{(1+\mu t)^{sa}t^\alpha\right\},\qquad \text{where}\quad \mu=\frac{\lambda}{1+\lambda a}$$

$$=e_\lambda^{-sa}\int_0^\infty(1+\mu t)^{\frac{-s}{\mu}}(1+\mu t)^{sa}t^\alpha dt,\qquad \text{from}\quad (1.11)$$

$$=e_\lambda^{-sa}\int_0^\infty(1+\mu t)^{\frac{-s(1-\mu a)}{\mu}}t^\alpha dt$$

which,

$$=e_\lambda^{-sa}\int_0^\infty\left(1+\frac{\mu}{s(1-\mu a)}s(1-\mu a)t\right)^{\frac{-s(1-\mu a)}{\mu}}\left\{\frac{s(1-\mu a)}{s(1-\mu a)}t\right\}^\alpha\frac{s(1-\mu a)}{s(1-\mu a)}dt$$

$$=\frac{e_\lambda^{-sa}}{s^{\alpha+1}(1-\mu a)^{\alpha+1}}\int_0^\infty\left(1+\frac{\mu}{s(1-\mu a)}y\right)^{\frac{-s(1-\mu a)}{\mu}}y^\alpha dy,\qquad \text{where}\ y=s(1-\mu a)t$$

on putting $\lambda=\dfrac{\mu}{1-\mu a}$ and taking $s(1-\mu a)>0$, with the help of (1.12) leads us to

$$\mathbf{L}_\lambda\left\{g(t)\right\}=\frac{e_\lambda^{-sa}(1+\lambda a)^{\alpha+1}}{s^{\alpha+1}}\Gamma_{\frac{\lambda}{s}}(\alpha+1)$$

Here it is to be noted that the result of the above equation, in the limiting case, when $\lambda\to 0+$ reduces to the expected result $e^{-as}\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, as is obtained by the use of the second translation theorem in the theory of the classical Laplace transform.

Theorem 2.3: Change of Scale Property: If $\mathbf{L}_\lambda\left\{f(t)\right\}=F_\lambda(s)$, then

$$\mathbf{L}_\lambda\left\{f(at)\right\}=\frac{1}{a}F_{\frac{\lambda}{a}}\left(\frac{s}{a}\right),\qquad a\neq 0 \tag{2.10}$$

Proof: We have

$$\mathbf{L}_\lambda\left\{f(at)\right\}=\int_0^\infty(1+\lambda t)^{\frac{-s}{\lambda}}f(at)dt$$

$$=\int_0^\infty\left(1+\frac{\lambda}{a}u\right)^{\frac{-s}{\lambda}}f(u).\frac{1}{a}du,\qquad \text{where}\ at=u$$

$$=\frac{1}{a}\int_0^\infty(1+\beta u)^{\frac{-s}{a\beta}}f(u)du,\qquad \text{where}\ \beta=\frac{\lambda}{a}$$

$$=\frac{1}{a}F_\beta\left(\frac{s}{a}\right)=\frac{1}{a}F_{\frac{\lambda}{a}}\left(\frac{s}{a}\right).$$

This theorem generalizes the Theorem 1-5, p.3 Chapter 1 of [7] or the Theorem 3.5, p.107 of [26] to which it reduces when $\lambda\to 0+$ in (2.10). As an illustration, the validity of the results in (2.2) and (2.4) can be checked in a straightforward manner by the use of the Theorem 2.3 by starting respectively with the results $\mathbf{L}_\lambda\left\{\cos_\lambda(t)\right\}=\dfrac{s-\lambda}{(s-\lambda)^2+1}$ and $\mathbf{L}_\lambda\left\{\sin_\lambda(t)\right\}=\dfrac{1}{(s-\lambda)^2+1}$.

We conclude the paper by mentioning that this author has already established the generalizations of many more results for the degenerate Laplace transforms of a function which have appeared very recently in the successive papers [27-29] of this series of papers.

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