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Application of the Upadhyaya transform to Volterra integral equations of the first kind *

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Abstract The application of the methods of integral transforms has tremendously increased during the past many years for finding the exact solutions and the numerical solutions of many new problems arising in the fields of engineering and applied sciences. Some two years back the Upadhyaya transform (https://www.researchgate.net/publication/334033797) was introduced as the most powerful generalization and the most powerful unification of a number of existing variants of the classical Laplace transform occurring in the mathematics research literature. In this paper we apply the Upadhyaya transform for solving the Volterra integral equations of the first kind. Our analysis leads to the conclusion that the Upadhyaya transform is a very powerful tool for finding the solutions to the Volterra integral of the first kind.

Key words Upadhyaya transform, Inverse Upadhyaya transform, Volterra integral equations, Convolution theorem, Bessel function.

2020 Mathematics Subject Classification 34A12, 44A05, 44A99, 45D05, 45D99.

1 Introduction

On account of the growing use of the methods of integral transforms during the past many years in finding the solutions to the complex problems arising in engineering, physics and the other branches of study including the biological sciences, nuclear sciences, social sciences, economics, etc. many new variants of the classical Laplace transform are introduced into the mathematics research literature during the past about three decades, like the Sumudu transform, the Elzaki transform, the Mahgoub transform, etc. A detailed and exhaustive description of many of these transforms can be found in Upadhyaya [17] and Upadhyaya et al. [18]. The Upadhyaya transform (UT) was introduced by Upadhyaya [17] about two and a half years back. We mention that the Upadhyaya transform is the most powerful generalization and the most versatile unification of almost all the classical variants of the Laplace transform that exist in the mathematics research literature today. This most general variant of the classical Laplace transform has the capability of application to any of the various diverse phenomenon for finding the solution to any problem to which any of the numerous variants of the classical Laplace transform have been applied earlier by different researchers during the past more than

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two decades. In this paper we show the utility and importance of the Upadhyaya transform (UT) in solving the linear Volterra integral equations of the first kind. For the sake of the interested reader we point out that the Volterra integral equations are well studied in the literature. As some relevant references, we mention just a few - Hendry [1], Ahmad and Tahir [2], Berenguer et al. [3], Babolian and Masouri [4], Armand and Gouyandeh [5], Brunner [6], Markova and Sidorov [7], Gnanavel et al. [8], Mirzaee [9], Davies and Duncan [10], Aggarwal et al. [11] , Ngarasta [12], Taylor [13], Sidorov et al. [14]. An excellent introductory work on Volterra integral equations is Brunner [15]. Aggarwal and his coworkers have extensively studied the Volterra integral equations of the first and second kinds and the Abel's integral equations with the help of a number of Laplace type integral transforms [19–36].
The Upadhyaya transform of the function $F(t), t \geq 0$ is given by [17]:

$$\mathcal{U}\left[F(t)\right] = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \int_0^\infty e^{-\lambda_2 t} F(\lambda_3 t) dt \;, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0 \tag{1.1}$$

where the operator $\mathcal{U}$ is called the Upadhyaya transform operator. Although in the general definition of the Upadhyaya transform, the parameters λ_1, λ_2 and λ_3 are complex parameters suitably defined (see Upadhyaya [17] and Upadhyaya et al. [18]), but for the purposes of this paper, it suffices for us to assume that all these three parameters are positive real numbers, therefore, we justify this choice of ours specifically in (1.1).
The Upadhyaya transform of a function $F(t), t \geq 0$ exists if $F(t)$ is piecewise continuous and of exponential order. These mentioned conditions are only the sufficient conditions for the existence of Upadhyaya transform of the function $F(t)$.
The Volterra integral equation of the first kind (e.g., see, Weisstein [16]) is given by

$$f(t) = \int_a^t k(t, x) g(x) dx \tag{1.2}$$

where the unknown function $g(x)$, that will be determined, occurs only inside the integral sign. The kernel $k(x, t)$ of integral equation (1.2) and the function $f(t)$ are known real-valued functions.
The main aim of our study here is to find out the exact solution of the Volterra integral equation of the first kind by using the most recent and the most advanced integral transform of the Laplace class - the "Upadhyaya transform" without involving a large computational work.

2 The linearity property of the Upadhyaya transform (UT)

If $F_1(t), F_2(t)$ be two functions with UTs $\mathfrak{u}_1(\lambda_1, \lambda_2, \lambda_3)$ and $\mathfrak{u}_2(\lambda_1, \lambda_2, \lambda_3)$ relative to the parameters $\lambda_1, \lambda_2, \lambda_3$ and c_1, c_2 be any constants then,

$$\begin{aligned}\mathcal{U}\left[c_1 F_1(t) + c_2 F_2(t); \lambda_1, \lambda_2, \lambda_3\right] &= c_1 \mathcal{U}\left[F_1(t); \lambda_1, \lambda_2, \lambda_3\right] + c_2 \mathcal{U}\left[F_2(t); \lambda_1, \lambda_2, \lambda_3\right] \\ &= c_1 \mathfrak{u}_1(\lambda_1, \lambda_2, \lambda_3) + c_2 \mathfrak{u}_2(\lambda_1, \lambda_2, \lambda_3)\end{aligned}$$

where c_1, c_2 are arbitrary constants.

3 The Upadhyaya transforms of some useful functions

In this section, for the convenience of the readers we tabulate above in Table 1 the UTs of some frequently used elementary functions, some of which will also find ready applications in our work later on in this paper also.

4 The inverse Upadhyaya transform

If$U\left[F(t)\right] = u(\lambda_1, \lambda_2, \lambda_3)$ then $F(t)$ is called the inverse Upadhyaya transform of $u(\lambda_1, \lambda_2, \lambda_3)$ and mathematically, it is defined as $F(t) = U^{-1}\left[u(\lambda_1, \lambda_2, \lambda_3)\right]$ where the operator U^{-1} is called the inverse Upadhyaya transform operator.

5 The inverse Upadhyaya transform of some useful functions

We list below in Table 2 the inverse Upadhyaya transforms of some commonly occurring functions. Some of these results will also be used by us in the applications section of this paper.

Table 1: The Upadhyaya transforms of some useful functions.

S.N.	$F(t)$	$\mathcal{U}\left[F(t);\lambda_1,\lambda_2,\lambda_3\right] = \mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$	S.N.	$F(t)$	$\mathcal{U}\left[F(t);\lambda_1,\lambda_2,\lambda_3\right] = \mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$
1.	1	$\frac{\lambda_1}{\lambda_2}$	7.	$\sin at$	$\frac{a\lambda_1\lambda_3}{\lambda_2^2+a^2\lambda_3^2}$
2.	t	$\frac{\lambda_1\lambda_3}{\lambda_2^2}$	8.	$\cos at$	$\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2}$
3.	t^2	$\frac{2\lambda_1\lambda_3^2}{\lambda_2^3}$	9.	$\sinh at$	$\frac{a\lambda_1\lambda_3}{\lambda_2^2-a^2\lambda_3^2}$
4.	t^n , $n \in \mathbb{N}$	$\frac{n!\lambda_1\lambda_3^n}{\lambda_2^{n+1}}$	10.	$\cosh at$	$\frac{\lambda_1\lambda_2}{\lambda_2^2-a^2\lambda_3^2}$
5.	t^n, $n > -1$	$\frac{\lambda_1\lambda_3^n\Gamma(n+1)}{\lambda_2^{n+1}}$	11.	$t^n e^{at}$, $n > -1$	$\frac{\lambda_1\lambda_3^n\Gamma(n+1)}{(\lambda_2-a\lambda_3)^{n+1}}$
6.	e^{at}	$\frac{\lambda_1}{\lambda_2-a\lambda_3}$	12.	$J_0(at)$	$\frac{\lambda_1}{\sqrt{\lambda_2^2+a^2\lambda_3^2}}$

Table 2: The inverse Upadhyaya transforms of some useful functions.

S.N.	$\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$	$F(t) = \mathcal{U}^{-1}\left[\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)\right]$	S.N.	$\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)$	$F(t) = \mathcal{U}^{-1}\left[\mathfrak{u}(\lambda_1,\lambda_2,\lambda_3)\right]$
1.	$\frac{\lambda_1}{\lambda_2}$	1	7.	$\frac{a\lambda_1\lambda_3}{\lambda_2^2+a^2\lambda_3^2}$	$\sin at$
2.	$\frac{\lambda_1\lambda_3}{\lambda_2^2}$	t	8.	$\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2}$	$\cos at$
3.	$\frac{2\lambda_1\lambda_3^2}{\lambda_2^3}$	t^2	9.	$\frac{a\lambda_1\lambda_3}{\lambda_2^2-a^2\lambda_3^2}$	$\sinh at$
4.	$\frac{n!\lambda_1\lambda_3^n}{\lambda_2^{n+1}}$	t^n , $n \in \mathbb{N}$	10.	$\frac{\lambda_1\lambda_2}{\lambda_2^2-a^2\lambda_3^2}$	$\cosh at$
5.	$\frac{\lambda_1\lambda_3^n\Gamma(n+1)}{\lambda_2^{n+1}}$	t^n , $n > -1$	11	$\frac{\lambda_1\lambda_3^n\Gamma(n+1)}{(\lambda_2-a\lambda_3)^{n+1}}$	$t^n e^{at}$, $n > -1$
6.	$\frac{\lambda_1}{\lambda_2-a\lambda_3}$	e^{at}	12	$\frac{\lambda_1}{\sqrt{\lambda_2^2+a^2\lambda_3^2}}$	$J_0(at)$

6 Upadhyaya transform of the derivatives of a function

If $\mathcal{U}\left[F(t)\right] = \mathfrak{u}(\lambda_1, \lambda_2, \lambda_3)$ then from [17]

1. $\mathcal{U}\left[F'(t); \lambda_1, \lambda_2, \lambda_3\right] = \frac{\lambda_2}{\lambda_3}\mathcal{U}\left[F(t); \lambda_1, \lambda_2, \lambda_3\right] - \frac{\lambda_1}{\lambda_3}F(0)$, or,
   $\mathcal{U}\left[F'(t); \lambda_1, \lambda_2, \lambda_3\right] = \frac{\lambda_1}{\lambda_3}\mathcal{U}\left[F(t); \lambda_1, \lambda_2, \lambda_3\right] - \frac{\lambda_1}{\lambda_3}F(0)$.
2. $\mathcal{U}\left[F''(t); \lambda_1, \lambda_2, \lambda_3\right] = \left(\frac{\lambda_2}{\lambda_3}\right)^2 \mathcal{U}\left[F(t); \lambda_1, \lambda_2, \lambda_3\right] - \frac{\lambda_1\lambda_2}{\lambda_3^2}F(0) - \frac{\lambda_1}{\lambda_3}F'(0)$.
3. $\mathcal{U}\left[F^n(t); \lambda_1, \lambda_2, \lambda_3\right] = \left(\frac{\lambda_2}{\lambda_3}\right)^n \mathcal{U}\left[F(t); \lambda_1, \lambda_2, \lambda_3\right] - \frac{\lambda_1\lambda_2^{n-1}}{\lambda_3^n}F(0) - \frac{\lambda_1\lambda_2^{n-2}}{\lambda_3^{n-1}}F'(0) - \frac{\lambda_1\lambda_2^{n-3}}{\lambda_3^{n-2}}F''(0) - \cdots - \frac{\lambda_1}{\lambda_3}F^{n-1}(0)$.

7 Upadhyaya transform of Bessel functions of order one

Bessel function of order one is denoted by $J_1(t)$ and it is given by the following relation

$$\frac{d}{dx}J_0(t) = -J_1(t) \tag{7.1}$$

where $J_0(t)$ is the Bessel function of order zero. Operating Upadhyaya transform on both sides of (7.1), we get (see also Upadhyaya et al. [18])

$$\mathcal{U}\left[\frac{dJ_0(t)}{dt}\right] = -\mathcal{U}\left[J_1(t)\right] \tag{7.2}$$

Applying the property, Upadhyaya transform of derivative of function, in (7.2), we have

$$\frac{\lambda_2}{\lambda_3}\mathcal{U}\left[J_0(t); \lambda_1, \lambda_2, \lambda_3\right] - \frac{\lambda_1}{\lambda_3}J_0(0) = -\mathcal{U}\left[J_1(t); \lambda_1, \lambda_2, \lambda_3\right]$$

$$\begin{aligned}
&\mathcal{U}\left[J_1(t); \lambda_1, \lambda_2, \lambda_3\right] = \tfrac{\lambda_1}{\lambda_3} - \tfrac{\lambda_2}{\lambda_3}\mathcal{U}\left[J_0(t); \lambda_1, \lambda_2, \lambda_3\right]\\
&\Rightarrow \mathcal{U}\left[J_1(t); \lambda_1, \lambda_2, \lambda_3\right] = \tfrac{\lambda_1}{\lambda_3} - \tfrac{\lambda_2}{\lambda_3}\left[\tfrac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]\\
&\Rightarrow \mathcal{U}\left[J_1(t); \lambda_1, \lambda_2, \lambda_3\right] = \tfrac{\lambda_1}{\lambda_3}\left[1 - \tfrac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]
\end{aligned}$$

8 Convolution property of the Upadhyaya transform

If the Upadhyaya transform of the functions $F_1(t)$ and $F_2(t)$ w.r.t. the parameters $\lambda_1, \lambda_2, \lambda_3$ are $\mathfrak{u}_1(\lambda_1, \lambda_2, \lambda_3)$ and $\mathfrak{u}_2(\lambda_1, \lambda_2, \lambda_3)$ respectively then the Upadhyaya transform of their convolution $F_1(t) * F_2(t)$ is given by

$$\mathcal{U}\left[F_1(t) * F_2(t); \lambda_1, \lambda_2, \lambda_3\right] = \tfrac{\lambda_3}{\lambda_1}\mathfrak{u}_1(\lambda_1, \lambda_2, \lambda) \cdot \mathfrak{u}_2(\lambda_1, \lambda_2, \lambda_3).$$

where $F_1(t) * F_2(t)$ is defined by

$$\begin{aligned}
F_1(t) * F_2(t) &= \textstyle\int_0^t F_1(t-x)F_2(x)dx\\
&= \textstyle\int_0^t F_1(x)F_2(t-x)dx\,.
\end{aligned}$$

Proof. See Upadhyaya ([17, p. 493]). □

9 Applications

We now propose to revisit some Volterra integral equations of the first kind, which were earlier discussed and solved by Aggarwal et al. [11] by the application of the Shehu transform. Below we solve some of these integral eqautions by applying the Upadhyaya transform.

Example 9.1. (Aggarwal et al. [11, (8), p. 440]) Consider the Volterra integral equation of the first kind

$$x = \int_0^x e^{(x-t)} h(t)dt. \tag{9.1}$$

Taking the Upadhyaya transform to both sides of (9.1), we have

$$\mathcal{U}\left[x;\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[\int_0^x e^{(x-t)} h(t)dt;\lambda_1,\lambda_2,\lambda_3\right]$$

Using the convolution property of the Upadhyaya transform in (9.1), we have

$$\begin{aligned}
&\tfrac{\lambda_1\lambda_3}{\lambda_2^2} = \tfrac{\lambda_3}{\lambda_1}\mathcal{U}\left[e^x;\lambda_1,\lambda_2,\lambda_3\right]\cdot\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \tfrac{\lambda_1\lambda_3}{\lambda_2^2} = \tfrac{\lambda_3}{\lambda_1}\left[\tfrac{\lambda_1}{\lambda_2-\lambda_3}\right]\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \tfrac{\lambda_1^2\lambda_3(\lambda_2-\lambda_3)}{\lambda_1\lambda_2^2\lambda_3}\\
&\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \tfrac{\lambda_1}{\lambda_2} - \tfrac{\lambda_1\lambda_3}{\lambda_2^2}\\
&h(x) = \mathcal{U}^{-1}\left[\tfrac{\lambda_1}{\lambda_2} - \tfrac{\lambda_1\lambda_3}{\lambda_2^2}\right]\\
&h(x) = 1 - x
\end{aligned}$$

which is the required exact solution of (9.1).

Example 9.2. (Aggarwal et al. [11, (12), p. 440]) Consider the Volterra integral equation of the first kind

$$\sin x = \int_0^x e^{(x-t)} h(t)dt. \tag{9.2}$$

Applying the Upadhyaya transform to both sides of (9.2), we have

$$\mathcal{U}\left[\sin x;\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[\int_0^x e^{(x-t)} h(t)dt;\lambda_1,\lambda_2,\lambda_3\right]$$

Using the convolution property of the Upadhyaya transform in (9.2), we have

$$\begin{aligned}
&\tfrac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2} = \tfrac{\lambda_3}{\lambda_1}\mathcal{U}\left[e^x;\lambda_1,\lambda_2,\lambda_3\right]\cdot\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \tfrac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2} = \tfrac{\lambda_3}{\lambda_1}\left[\tfrac{\lambda_1}{\lambda_2-\lambda_3}\right]\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \tfrac{\lambda_1^2\lambda_3(\lambda_2-\lambda_3)}{\lambda_1\lambda_3\left(\lambda_2^2+\lambda_3^2\right)}\\
&\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \tfrac{\lambda_1\lambda_2}{\left(\lambda_2^2+\lambda_3^2\right)} - \tfrac{\lambda_1\lambda_3}{\left(\lambda_2^2+\lambda_3^2\right)}\\
&h(x) = \mathcal{U}^{-1}\left[\tfrac{\lambda_1\lambda_2}{\lambda_2^2+\lambda_3^2}\right] - \mathcal{U}^{-1}\left[\tfrac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2}\right]\\
&h(x) = \cos x - \sin x
\end{aligned}$$

which is the required exact solution of (9.2).

Example 9.3. (Aggarwal et al. [11, (16), p. 440]) Consider the Volterra integral equation of thefirst kind

$$\sin x = \int_0^x J_0(x-t) h(t)dt. \tag{9.3}$$

Applying the Upadhyaya transform to both sides of (9.3), we have

$$\mathcal{U}\left[\sin x;\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[\int_0^x J_0(x-t) h(t)dt;\lambda_1,\lambda_2,\lambda_3\right]$$

Using the convolution property of the Upadhyaya transform in (9.3), we have

$$\begin{aligned}
&\tfrac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2} = \tfrac{\lambda_3}{\lambda_1}\mathcal{U}\left[J_0(x);\lambda_1,\lambda_2,\lambda_3\right]\cdot\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \tfrac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2} = \tfrac{\lambda_3}{\lambda_1}\left[\tfrac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]\\
&\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \tfrac{\lambda_1\sqrt{\lambda_2^2+\lambda_3^2}}{\left(\lambda_2^2+\lambda_3^2\right)}
\end{aligned}$$

$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}$$
$$h(x) = \mathcal{U}^{-1}\left[\frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]$$
$$h(x) = J_0(x)$$

which is the required exact solution of (9.3).

Example 9.4. (Aggarwal et al. [11, (36), p. 441]) Consider the Volterra integral equation of the first kind

$$1 - J_0(x) = \int_0^x h(t)dt. \tag{9.4}$$

Applying the Upadhyaya transform to both sides of (9.4), we have

$$\mathcal{U}\left[1 - J_0(x);\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[\int_0^x h(t)dt;\lambda_1,\lambda_2,\lambda_3\right]$$
$$\mathcal{U}\left[1;\lambda_1,\lambda_2,\lambda_3\right] - \mathcal{U}\left[J_0(x);\lambda_1,\lambda_2,\lambda_3\right] = \mathcal{U}\left[1;\lambda_1,\lambda_2,\lambda_3\right]\cdot\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$

Using the convolution property of Upadhyaya transform in (9.4), we have

$$\frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}} = \frac{\lambda_3}{\lambda_1}\left[\frac{\lambda_1}{\lambda_2}\right]\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$
$$\Rightarrow \frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}} = \frac{\lambda_3}{\lambda_2}\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$
$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_2}{\lambda_3}\left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right)$$
$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1}{\lambda_3}\left(1 - \frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\right)$$
$$h(x) = \mathcal{U}^{-1}\left[\frac{\lambda_1}{\lambda_3}\left(1 - \frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\right)\right]$$
$$h(x) = J_1(x)$$

which is the required exact solution of (9.4).

Example 9.5. (Aggarwal et al. [11, (44), p. 442]) Consider the Volterra integral equation of the first kind

$$\cos x - J_0(x) = -\int_0^x J_1(x-t)h(t)dt. \tag{9.5}$$

Applying the Upadhyaya transform to both sides of (9.5), we have

$$\mathcal{U}\left[\cos x - J_0(x);\lambda_1,\lambda_2,\lambda_3\right] = -\mathcal{U}\left[\int_0^x J_1(x-t)h(t)dt;\lambda_1,\lambda_2,\lambda_3\right]$$
$$\mathcal{U}\left[\cos x;\lambda_1,\lambda_2,\lambda_3\right] - \mathcal{U}\left[J_0(x);\lambda_1,\lambda_2,\lambda_3\right] = -\mathcal{U}\left[J_1(x);\lambda_1,\lambda_2,\lambda_3\right]\cdot\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$

Using the convolution property of the Upadhyaya transform in (9.5), we have

$$\frac{\lambda_1\lambda_2}{\lambda_2^2+\lambda_3^2} - \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}} = -\frac{\lambda_3}{\lambda_1}\left[\frac{\lambda_1}{\lambda_3}\left(1 - \frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\right)\right]\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$
$$\Rightarrow \frac{\lambda_1\lambda_2}{\lambda_2^2+\lambda_3^2} - \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}} = \left(\frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}} - 1\right)\mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right]$$
$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \left(\frac{\lambda_1\lambda_2}{\lambda_2-\sqrt{\lambda_2^2+\lambda_3^2}}\right)\left(\frac{1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right) - \left(\frac{\lambda_1}{\lambda_2-\sqrt{\lambda_2^2+\lambda_3^2}}\right)$$
$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}$$

$$\Rightarrow \mathcal{U}\left[h(x);\lambda_1,\lambda_2,\lambda_3\right] = \frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}$$
$$h(x) = \mathcal{U}^{-1}\left[\frac{\lambda_1}{\sqrt{\lambda_2^2+\lambda_3^2}}\right]$$
$$h(x) = J_0(x)$$

which is the required exact solution of (9.5).

10 Conclusion

With the discussion as above in section 9 we have successfully demonstrated the efficacy and the elegance of the Upadhyaya transform as a versatile tool for solving the Volterra integral equations of the first kind, we can similarly solve the Volterra integral equations of the second kind by utilizing the various properties of the Upadhyaya transform. In fact this most powerful generalization of the classical Laplace transform can be applied to any phenomenon where the other variants of the classical Laplace transforms have earlier been applied by the numerous research workers and we propose to discuss the diverse applications of the Upadhyaya transform in our forthcoming papers.

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