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Note on the Elzaki Transform and the Martínez–Kaabar Volterra Integral Equations of the Second Kind

Adil Mousa [1] and Tarig M. Elzaki[2]

Author Affiliation:
[1]Department of Mathematics, Almanhal Academy of Science , Khartoum , Sudan
[2]Mathematics Department, Faculty of Sciences and Arts-Alkamil, University of Jeddah, Jeddah Saudi Arabia E-mail: aljarada@gmail.com[1], tarig.alzaki@gmail.com[2]

***Corresponding Author: Adil Mousa,** Department of Mathematics, Almanhal Academy of Science, Khartoum, Sudan
E-mail: aljarada@gmail.com

ABSTRACT

We have discussed and demonstrated the various properties and theorems of the Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform in this work, including the linearity property, the convolution theorem property, and the example application to solve the Martínez–Kaabar MK Volterra Integral Equation of the Second Kind.

Keywords: Fractal–Fractional, Fractal–Fractional Elzaki transform, Volterra integral equations

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1. INTRODUCTION

Integrated transformations have been used extensively as a method to address a variety of issues in both pure and practical mathematics during the last 200 years. The Laplace transform concept was first proposed in 1782 by a number of integrated transformations, including the P. S. Laplace (1749–1827). [3, 8],

$$L\left[g(\tau):\delta\right]=\int_0^\infty e^{-\delta\tau}g(\tau)d\tau=G(\delta)\ ,\ \tau>0 \qquad (1)$$

Tarig Elzaki first presented the modified Laplace transform (also known as the Elzaki transform) in the early 2011s.The function of exponential order is defined by the modified Laplace transform (also known as the Elzaki transform) [1, 2]. Examine the following function in the set S.

$$S=\{g(\tau):\exists M,k_1,k_2>0,\left|g(\tau)\right|<M\,e^{\frac{|\tau|}{k_j}},if\ \ \tau\in(-1)^j\times[0,\infty)\}$$

For a given function $g(\tau)$ in the set S, the constant M must be finite, number k_1, k_2 may be finite or infinite. The Elzaki transform denoted by the operator E is defined as

$$E\left[g(\tau):\rho)\right]=\rho\int_0^\infty e^{-\frac{\tau}{\rho}}g(\tau)d\tau=T(\rho), \quad \tau>0 \tag{2}$$

The variable ρ in this transforms is used to factorize the variable τ.

The following is how this document is structured: The Martinez–Kaabar fractal–fractional (MK FrFr) Calculus is first defined in some fundamental terms. Next, we establish a novel concept of the Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform, which in this work is called the MK Elzaki transform and involves the MK integral operator. The Martínez–Kaabar MK Volterra Integral Equation of the Second Kind may be solved using this novel approach, which is related to the new fractal–fractional transformation, the linearity property, and the Convolution theorem property.

2. FUNDAMENTAL CONCEPTS OF MARTINEZ–KAABAR FRACTAL–FRACTIONAL (MK FRFR) CALCULUS

Definition 2.1. [14] Suppose that $g(\tau)\in C^n\left(\left[\lambda,\infty\right)\right)$ is differentiable on $\left[\lambda,\infty\right)$, where $\lambda\geq 0$; if g is a fractal differentiable on $\left[\lambda,\infty\right)$ of order β, then the FrFrD of g of order α in the context of C with the power law is written as

$${}^{FrFrD}_{\ \lambda}D^{\alpha,\beta}_{\tau}g(\tau)=\frac{1}{\Gamma(m-\alpha)}\int_\lambda^\tau(\tau-x)^{m-\alpha-1}\frac{dg(x)}{dx^\beta}dx \ ,$$

(3)

$m-1<\alpha,\beta\leq m \ , m\in N$,

Where $$\frac{dg(x)}{dx^\beta}=\lim_{\tau\to x}\frac{g(\tau)-g(x)}{\tau^\beta-x^\beta} \ . \tag{4}$$

Theorem 2.1. Suppose that $0<\alpha,\beta\leq 1$, and $\sigma>-1$. Then, we have:

$${}^{FrFrD}_{\ 0}D^{\alpha,\beta}_{\tau}g(\tau^\sigma)=\frac{\sigma\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}\tau^{\sigma-\alpha-\beta+1} . \tag{5}$$

Remark 2.1. If $g(\tau)=\lambda$ for every real constant λ, then ${}^{FrFrD}_{\ 0}D^{\alpha,\beta}_{\tau}(\lambda)=0$.

Definition 2.2. A function: $g:\left[0,\infty\right)\to R$, the MK derivative of order $0<\alpha\leq 1$, of g at $\tau>0$ is written as

$${}^{MK}D^{\alpha,\beta}g(\tau)=\lim_{\delta\to 0}\frac{g\left(\tau+\delta\Psi\tau^{2-\alpha-\beta}\right)-g(\tau)}{\delta} \ , \tag{6}$$

Where $M\left(\alpha,\beta,\sigma\right)=\dfrac{\Gamma\left(\sigma-\beta+1\right)}{\beta\Gamma\left(\sigma-\alpha-\beta+2\right)}=\Psi$ with $0<\beta\leq 1$, and $\sigma>-1$.

If g is MK α,β-differentiable in some $\left(0,n\right)$, and $n>0$, and $\lim\limits_{\tau\to 0^+}{}^{MK}D^{\alpha,\beta}g(\tau)$ exists, then it is written as

$${}^{MK}D^{\alpha,\beta}g(0)=\lim_{\tau\to 0^+}{}^{MK}D^{\alpha,\beta}g(\tau) . \tag{7}$$

Theorem 2.2. [14] Suppose that $\alpha<0,\beta\leq 1$, , and g is a MK α,β-differentiable at a point $\tau>0$. If, further, g is a differentiable function, then

$${}^{MK}D^{\alpha,\beta}g(\tau)=\Psi\,{}^{MK}D^{\alpha,\beta}\tau^{2-\alpha-\beta}\frac{dg(\tau)}{d\tau} \ , \tag{8}$$

Where $\Psi=M\left(\alpha,\beta,\sigma\right)=\dfrac{\Gamma\left(\sigma-\beta+1\right)}{\beta\Gamma\left(\sigma-\alpha-\beta+2\right)}$ with $\sigma>-1$.

Theorem 2.3. (Chain Rule). Assume that $\alpha<0,\beta\leq 1$, $\sigma>-1, f$ is an MK α,β-differentiable at $\tau > 0$ and g is differentiable at $f(\tau)$, then

$$ {}^{MK}D^{\alpha,\beta}(g \circ f)(\tau) = g'(f(\tau))\, {}^{MK}D^{\alpha,\beta} f(\tau). $$

(9)

Remark 2.2. According to Theorem 2.6, the MK derivative of order α of some elementary functions can be expressed as:

i- $ {}^{MK}D^{\alpha,\beta}\left[\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}\right] = 1 $.

ii- $ {}^{MK}D^{\alpha,\beta}\left[e^{\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}}\right] = e^{\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}} $

iii- $ {}^{MK}D^{\alpha,\beta}\left[\sin\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right)\right] = \cos\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right) $.

iv- $ {}^{MK}D^{\alpha,\beta}\left[\cos\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right)\right] = -\sin\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right) $.

Remark 2.3. From the differentiability property of the MK derivative, which is α,β-differentiability, and by assuming that $g(\tau) > 0$, then Equation (9) can be represented as:

$$ {}^{MK}D^{\alpha,\beta}(f \circ g)(\tau) = \frac{1}{\Psi} g(\tau)^{\alpha+\beta-2}\, {}^{MK}D^{\alpha,\beta} f(g(\tau))\, {}^{MK}D^{\alpha,\beta} g(\tau). $$

(10)

Where $\Psi = M(\alpha,\beta,\sigma) = \frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$ with $\sigma > -1$.

The MK α,β-integral of a function g starting at $\lambda \ge 0$, can be recalled as formulated in [9].

Definition 2.3. $ {}^{MK}I^{\lambda}_{\alpha,\beta}(g)(\tau) = \frac{1}{\Psi}\int_{\lambda}^{\tau}\frac{g(t)}{t^{2-\alpha-\beta}}\cdot dt $,such that this integral is the well-known Riemann improper integral, $\Psi = M(\alpha,\beta,\sigma) = \frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, with $0 < \beta \le 1$, and $\sigma > -1$.

From Definition 2.3, we obtain the following:

Theorem 2.4. $ {}^{MK}D^{\alpha,\beta}\, {}^{MK}I^{\lambda}_{\alpha,\beta}(g)(\tau) = g(\tau)$, for $\tau \ge n$,, such that g is any continuous function in the domain of ${}^{MK}I^{\lambda}_{\alpha,\beta}$.

Theorem 2.5. Suppose that $\lambda > 0$, $0 < \alpha$, $\beta \le 1$, $\sigma > -1$, and g is a continuous real-valued function (RVF) on $[\lambda,\kappa]$ Let V be any RVF with the property: ${}^{MK}D^{\alpha,\beta}V(\tau) = g(\tau)$, for all $\tau \in [\lambda,\kappa]$.

Then

$$ {}^{MK}I^{\lambda}_{\alpha,\beta}(g)(\kappa) = V(\kappa) - V(\lambda). \tag{11} $$

The goals of our study with respect to the concept of the MK Volterra integral equation, which was put forward in [17], must also be mentioned. It is therefore often provided as follows:

$$ V(\tau)W(\tau) = g(\tau) + \frac{\Omega}{\Psi}\int_0^{\tau} V(\tau,x)W(x)\frac{dx}{x^{2-\alpha-\beta}}, \tag{12} $$

Where $V(\tau,x)$ is the integral equation's kernel, $W(x)$ is an unknown function, $g(\tau)$ is a perturbation known function, Ω is non-zero real parameter, $\Psi = M(\alpha,\beta,\sigma) = \frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, with $0 < \beta \le 1$, and $\sigma > -1$.

However, if $V(\tau) = 1$, then Equation (12) is expressed as:

$$W(\tau) = g(\tau) + \frac{\Omega}{\Psi}\int_0^{\tau} V(\tau, x) W(x)\frac{dx}{x^{2-\alpha-\beta}},$$

which is the Volterra integral equation of the second kind.

3. THEOREMS AND PROPERTIES OF FRACTAL–FRACTIONAL ELZAKI TRANSFORM

A novel concept of fractal–fractional Elzaki transform was defined in this section; it is called the Martínez–Kaabar fractal–fractional Elzaki transform in this study. From here on, we shall refer to it as the MK Elzaki transform in all future findings. The MK Elzaki transform was similarly defined, and the following attributes were listed:

Definition:3.1 Let that $0 < \alpha, \beta \le 1$, $\sigma > -1$,, and $g : [0, \infty) \to R$ is an RVF. Then, the MK Elzaki transform of order α, β is known as

$$E_{MK}\left[g(\tau):\rho)\right] = \frac{\rho}{\Psi}\int_0^{\infty} e^{-\frac{1}{\Psi}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}} g(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}} = T_{\alpha,\beta}(\rho). \tag{13}$$

provided that integral exists.

Theorem: 3.1. Let $0 < \alpha, \beta \le 1$, with $\alpha + \beta - 1 > 0$, $\sigma > -1$, and $\omega \in R, \rho > 0$, $\Psi = M(\alpha, \beta, \sigma) = \dfrac{\Gamma(\sigma - \beta + 1)}{\beta\Gamma(\sigma - \alpha - \beta + 2)}$, So, we have

i- If $f(\tau) = \omega$ then $E_{MK}\left[\omega:\rho)\right] = \omega\rho^2$

ii- If $f(\tau) = \tau^j$ then $E_{MK}\left[\tau^j:\rho)\right] = \rho^{\left[2+\frac{j}{(\alpha+\beta-1)}\right]}\left[\Psi(\alpha+\beta-1)\right]^{\frac{j}{(\alpha+\beta-1)}}\Gamma\left(1+\frac{j}{(\alpha+\beta-1)}\right)$

iii- If $f(\tau) = e^{\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}$ then $E_{MK}\left[e^{\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}:\rho\right] = \dfrac{\rho^2}{1-\omega\rho}$

iv- If $f(\tau) = \sin\left(\dfrac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right)$ then $E_{MK}\left[\sin\left(\dfrac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right):\rho\right] = \dfrac{\omega\rho^3}{1+\omega^2\rho^2}$

v- If $f(\tau) = \cos\left(\dfrac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right)$ then $E_{MK}\left[\cos\left(\dfrac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right):\rho\right] = \dfrac{\rho^2}{1+\omega^2\rho^2}$

Proof:

i- $$E_{MK}\left[\omega:\rho)\right] = \frac{\rho}{\Psi}\int_0^{\infty} e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}(\omega)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$= \omega\rho\left[-\rho e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\right]_0^{\infty} = \omega\rho^2$$

ii $$E_{MK}\left[\tau^j:\rho)\right] = \frac{\rho}{\Psi}\int_0^{\infty} e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\left(\tau^j\right)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$= \frac{\rho}{\Psi}\int_0^{\infty} e^{-\frac{\tau^{(\alpha+\beta-1)+j}}{\rho\Psi(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\left[\Psi\left(\alpha+\beta-1\right)\right]^{\frac{j}{(\alpha+\beta-1)}}\rho}{\Psi}\int_0^\infty e^{-\frac{\tau^{(\alpha+\beta-1)+j}}{\rho\Psi(\alpha+\beta-1)}}\left[\Psi\left(\alpha+\beta-1\right)\right]^{\frac{-j}{(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\left[\Psi\left(\alpha+\beta-1\right)\right]^{\frac{j}{(\alpha+\beta-1)}}\rho^{2+\frac{j}{\alpha+\beta-1}}\Gamma\left(1+\tfrac{j}{\alpha+\beta-1}\right)$$

iii- $E_{MK}\left[e^{\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}:\rho\right]=\frac{\rho}{\Psi}\int_0^\infty e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\left(e^{\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}\right)\frac{d\tau}{\tau^{2-\alpha-\beta}}$

$$=\frac{\rho}{\Psi}\int_0^\infty e^{-\frac{\tau^{\alpha+\beta-1}(1-\omega\rho)}{\rho\Psi(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\rho\left[-\frac{\rho}{1-\omega\rho}e^{-\frac{\tau^{\alpha+\beta-1}(1-\omega\rho)}{\rho\Psi(\alpha+\beta-1)}}\right]_0^\infty=\frac{\rho^2}{1-\omega\rho}$$

iv- $E_{MK}\left[\sin\left(\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi\left(\alpha+\beta-1\right)}\right):\rho\right]=E_{MK}\left[\frac{e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}-e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}}{2i}\right]$

$$=\frac{1}{2i}\left(E_{MK}\left[e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}\right]-\left[e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}\right]\right)=\frac{1}{2i}\left(\frac{\rho^2}{1-i\,\omega\rho}-\frac{\rho^2}{1+i\,\omega\rho}\right)=\frac{\omega\rho^3}{1+\omega^2\rho^2}$$

v- $E_{MK}\left[\cos\left(\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi\left(\alpha+\beta-1\right)}\right):\rho\right]=E_{MK}\left[\frac{e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}+e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}}{2}\right]$

$$=\frac{1}{2}\left(E_{MK}\left[e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}\right]+\left[e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}\right]\right)=\frac{1}{2}\left(\frac{\rho^2}{1-i\,\omega\rho}+\frac{\rho^2}{1+i\,\omega\rho}\right)=\frac{\rho^2}{1+\omega^2\rho^2}$$

Theorem: 3.2. (linearity of the MK Elzaki transform)
Let that $0<\alpha\ ,\beta\le 1$, , with $\alpha+\beta-1>0$, $\sigma>-1$, $f\ :[0,\infty)\to R$ are RVFs, and $\theta_1,\theta_2\in R$. If $E_{MK}\left[f\,(\tau)\right](\rho)$ and $E_{MK}\left[g(\tau)\right](\rho)$ exist , and $\Psi=M\,\left(\alpha,\beta,\sigma\right)=\frac{\Gamma\left(\sigma-\beta+1\right)}{\beta\Gamma\left(\sigma-\alpha-\beta+2\right)}$, then

$$E_{MK}\left[\theta_1 f\,(\tau)+\theta_2 g(\tau)\right](\rho)=\theta_1 E_{MK}\left[f\,(\tau)\right](\rho)+\theta_2 E_{MK}\left[g(\tau)\right](\rho)$$

Where q_1 and q_2 are constants
Proof:

$$E_{MK}\left[\theta_1 f\,(\tau)+\theta_2 g(\tau)\right](\rho)=\frac{\rho}{\Psi}\int_0^\infty\left[\theta_1 f\,(\tau)+\theta_2 g(\tau)\right]e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{\Psi}\int_0^\infty\left[\theta_1 f\,(\tau)\right]e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}+\frac{\rho}{\Psi}\int_0^\infty\left[\theta_2 g(\tau)\right]e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\theta_1\,\rho}{\Psi}\int_0^\infty e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}f\,(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}+\frac{\theta_2\,\rho}{M\,\left(\alpha,\beta,\sigma\right)}\int_0^\infty e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\theta_1 E_{MK}\left[f\,(\tau)\right](\rho)+\theta_2 E_{MK}\left[g(\tau)\right](\rho)$$

Theorem: 3.3 (Convolution theorem)
Let that $0<\alpha\ ,\beta\le 1$, , with $\alpha+\beta-1>0$, $\sigma>-1$, $f:[0,\infty)\to R$ are RVFs, and If $E_{MK}\left[f\,(\tau^{\alpha+\beta-1})\,\right](\rho)$ and $E_{MK}\left[g(\tau)\,\right](\rho)$ exist, and $\Psi=M\,(\alpha,\beta,\sigma)=\dfrac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, then we have

$$E_{MK}\left[f*g\,\right](\rho)=\frac{1}{\rho}E_{MK}\left[f\,(\tau^{\alpha+\beta-1})\right](\rho).E_{MK}\left[g(\tau)\right](\rho).$$
(14)

Where $$(f*g)(\tau)=\frac{1}{\Psi}\int_0^{\tau}f\,(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})g(x)\frac{dx}{x^{2-\alpha-\beta}}.$$
(15)

Proof: By taking MK Elzaki transform to Eq. (13), we have:

$$E_{MK}\left[f*g\,\right](\rho)=\frac{\rho}{\Psi}\int_0^{\infty}e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\left[\frac{1}{\Psi}\int_0^{\tau}f\,(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})g(x)\frac{dx}{x^{2-\alpha-\beta}}\right]\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{\Psi}\int_0^{\infty}e^{-\frac{x^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(x)\left[\frac{1}{\Psi}\int_x^{\infty}e^{-\frac{(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})}{\rho\Psi(\alpha+\beta-1)}}f\,(\frac{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}{\alpha+\beta-1})\frac{d\tau}{\tau^{2-\alpha-\beta}}\right]\frac{dx}{x^{2-\alpha-\beta}}$$

Now, consider the change of variables, $\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}=\nu^{\alpha+\beta-1}$

$$E_{MK}\left[f*g\,\right](\rho)=\frac{\rho}{\Psi}\int_0^{\infty}e^{-\frac{x^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(x)\left[\frac{1}{\Psi}\int_x^{\infty}e^{-\frac{\nu^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}f\,(\nu^{\alpha+\beta-1})\frac{d\nu}{\nu^{2-\alpha-\beta}}\right]\frac{dx}{x^{2-\alpha-\beta}}$$

$$=\frac{1}{\rho}E_{MK}\left[f\,(\tau^{\alpha+\beta-1})\right](\rho).E_{MK}\left[g(\tau)\right](\rho)$$

Theorem: 3.4. Suppose that $g(\tau)$ is continuous and ${}^{MK}D^{\alpha,\beta}g(\tau)$ is piece-wise continuous for all $\tau>0$. Then, $E_{MK}\left[{}^{MK}D^{\alpha,\beta}g(\tau)\right](\rho)$ exists , and $\Psi=M\,(\alpha,\beta,\sigma)=\dfrac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, and we have,

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}g(\tau):\rho\right]=\frac{1}{\rho}T_{\alpha,\beta}(\rho)-\rho g(0)$$

Proof: By taking Equation (13), we so

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}g(\tau):\rho\right]=\frac{\rho}{\Psi}\int_0^{\infty}e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}\left[{}^{MK}D^{\alpha,\beta}g(\tau)\right]\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

Applying the integration by parts, we have,

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}g(\tau):\rho\right]=\rho\left[e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(\tau)\right]_0^{\infty}+\frac{1}{\Psi}\int_0^{\infty}e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\rho\left[0-f\,(0)\right]+\frac{1}{\Psi}\int_0^{\infty}e^{-\frac{\tau^{\alpha+\beta-1}}{\rho\Psi(\alpha+\beta-1)}}g(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{1}{\rho}T_{\alpha,\beta}(\rho)-\rho g(0)$$

Corollary: 3.1. Assume that $f\,(\tau),{}^{MK}D^{\alpha,\beta}g(\tau),....,{}^{MK}D^{\alpha,\beta}_{(m-1)}g(\tau)$ are continuous, and ${}^{MK}D^{\alpha,\beta}_{(m)}g(\tau)$ is piece-wise continuous for all $\tau>0$. Suppose further that $g(\tau),{}^{MK}D^{\alpha,\beta}g(\tau),....,{}^{MK}D^{\alpha,\beta}_{(m-1)}g(\tau)$ are GFrFrEO. Then, $E_{MK}\left[{}^{MK}D^{\alpha,\beta}g(\tau)\right](\rho)$ exists , is given by:

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}_{(m)}g(\tau):\rho\right]=\frac{1}{\rho^m}T_{\alpha,\beta}(\rho)-\sum_{k=0}^{m-1}\rho^{2-m+k}\ {}^{MK}D^{\alpha,\beta}_{(k)}g(0)$$

Hence, ${}^{MK}D^{\alpha,\beta}_{(m)}g(\tau)$ means the application of the MK α,β – derivative m times.

Theorem: 3.5. Assume that g is a GFrFrEO and continuous for $\tau \geq 0$. Then, we obtain,

$$E_{MK}\left[\frac{1}{\Psi}\int_0^\tau f(x)\frac{dx}{x^{2-\alpha-\beta}}:\rho\right]=\rho E_{MK}\left[f(x):\rho\right]$$

Proof: By taking Theorem 3.3, If we take $g(\tau)=1$ and $E_{MK}\left[(1):\rho\right]=\rho^2$ from Theorem 3.1, our result follows easily.

4. APPLICATIONS

In this section, the Martínez–Kaabar MK Volterra Integral Equation of the Second Kind may be solved using the potent classical method of the MK Elzaki transform [17].

Consider the MK Volterra Integral Equation of the second kind:

$$W(\tau)=g(\tau)+\frac{\Omega}{\Psi}\int_0^\tau V(\tau,x)W(x)\frac{dx}{x^{2-\alpha-\beta}},$$

where $V(\tau,x)$ is the kernel , $W(x)$ is an unknown function , $g(\tau)$ is a perturbation known function, Ω is a non-zero real parameter, and $V(\tau,x)=V\left[\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right]$, and $\Psi=M(\alpha,\beta,\sigma)=\frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, then it is named the α,β -difference kernel. So, we have

$$W(\tau)=g(\tau)+\Omega[V*W](\tau)=g(\tau)+\frac{\Omega}{\Psi}\int_0^\tau V(\tau,x)W(x)\frac{dx}{x^{2-\alpha-\beta}},$$

(16)

By using the MK Elzaki transform on both sides of Equation (16), by taking Theorem 3.5 we get

$$E_{MK}\left[W(\tau)\right](\rho)\left(\frac{\rho-\Omega E_{MK}\left[V(\tau^{\alpha+\beta-1})\right](\rho)}{\rho}\right)=E_{MK}\left[g(\tau)\right](\rho)$$

$$E_{MK}\left[W(\tau)\right](\rho)=\left(\frac{\rho E_{MK}\left[g(\tau)\right](\rho)}{\rho-\Omega E_{MK}\left[V(\tau^{\alpha+\beta-1})\right](\rho)}\right)$$

A using inverse MK Elzaki transform, we obtain

$$W(\tau)=E^{-1}_{MK}\left(\frac{\rho E_{MK}\left[g(\tau)\right](\rho)}{\rho-\Omega E_{MK}\left[V(\tau^{\alpha+\beta-1})\right](\rho)}:\tau\right) \quad (17)$$

Problem 1: Consider the MK FrFr Volterra Integral Equation of the second kind:

$$W(\tau)=\frac{(2\tau)^{5-2\alpha-2\beta}}{\Psi(5-2\alpha-2\beta)}+\frac{4}{(3-\alpha-\beta)\Psi^2}\int_0^\tau\left(\tau^{3-\alpha-\beta}-x^{3-\alpha-\beta}\right)W(x)\frac{dx}{x^{2-\alpha-\beta}}.$$

(18)

Solution:

Appling MK Elzaki transform Eq. (18)

$$W(\tau)=\frac{(2\tau)^{5-2\alpha-2\beta}}{\Psi(5-2\alpha-2\beta)}+\frac{4}{(3-\alpha-\beta)\Psi^2}\int_0^\tau\left(x^{3-\alpha-\beta}-\tau^{3-\alpha-\beta}\right)W(x)\frac{dx}{x^{2-\alpha-\beta}}.$$

$$E_{MK}\left[W(\tau)\right](\rho)=\rho^3\Gamma(2)-4\rho^2.E_{MK}\left[W(\tau)\right](\rho)$$

Taking inverse MK Elzaki transform, we get

$$W(\tau)=\sin\left[\frac{(2\tau)^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right]$$

Problem 2: Consider the MK FrFr Volterra Integral Equation of the second kind:
$$W(\tau)=\frac{\tau^{3(\alpha+\beta-1)}}{\Gamma(3(\alpha+\beta)-2)}-\frac{1}{\Psi}\int_0^\tau(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})W(x)\frac{dx}{x^{2-\alpha-\beta}}.$$
(19)

Solution:

Taking MK Elzaki transform Eq.(19)

$$E_{MK}[W(\tau)](\rho)=\frac{[\Psi(\alpha+\beta-1)]^3\rho^5\Gamma(4)}{\Gamma(3(\alpha+\beta)-2)}-\Psi(\alpha+\beta-1)\rho^2.E_{MK}[W(\tau)](\rho)$$

$$E_{MK}[W(\tau)](\rho)\left[1+\rho^2\Psi(\alpha+\beta-1)\right]=\frac{[\Psi(\alpha+\beta-1)]^3\rho^5\Gamma(4)}{\Gamma(3(\alpha+\beta)-2)}$$

$$E_{MK}[W(\tau)](\rho)=\frac{[\Psi(\alpha+\beta-1)]^3\rho^5\Gamma(4)}{\Gamma(3(\alpha+\beta)-2)}.\frac{1}{\left[1+\rho^2M(\alpha,\beta,\sigma)(\alpha+\beta-1)\right]}$$

$$E_{MK}[W(\tau)](\rho)=\frac{[\Psi(\alpha+\beta-1)]^2\Gamma(4)}{\Gamma(3(\alpha+\beta)-2)}.\left(\rho^3-\frac{\rho^3}{\left[1+\rho^2\Psi(\alpha+\beta-1)\right]}\right)$$

Taking inverse MK Elzaki transform, we have

$$W(\tau)=\frac{\Psi(\alpha+\beta-1)\Gamma(4)}{\Gamma(3(\alpha+\beta)-2)}\tau^{\alpha+\beta-1}-\frac{\Gamma(4)[\Psi(\alpha+\beta-1)]^{\frac{3}{2}}}{\Gamma(3(\alpha+\beta)-2)}\sin\left(\sqrt{\Psi(\alpha+\beta-1)}\tau^{\alpha+\beta-1}\right)$$

Problem 3: Consider the MK FrFr Volterra Integral Equation of the second kind:
$$W(\tau)=-1-\frac{3^{\alpha+\beta-1}}{\Psi}\int_0^\tau W(x)\frac{dx}{x^{2-\alpha-\beta}}.$$
(20)

Solution:

Applying MK Elzaki transform Eq.(20)

$$E_{MK}[W(\tau)](\rho)=-\rho^2-\frac{1}{\rho}3^{\alpha+\beta-1}\Psi(\alpha+\beta-1)\rho^2.E_{MK}[W(\tau)](\rho)$$

$$E_{MK}[W(\tau)](\rho)\left[1+\rho 3^{\alpha+\beta-1}\Psi(\alpha+\beta-1)\right]=-\rho^2$$

$$E_{MK}[W(\tau)](\rho)=\frac{-\rho^2}{1+\rho 3^{\alpha+\beta-1}\Psi(\alpha+\beta-1)}$$

Taking inverse MK Elzaki transform, we get

$$W(\tau)=-e^{\frac{-(3\tau)^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}$$

Problem 4: Consider the MK FrFr Volterra Integral Equation of the second kind:
$$W(\tau)=\sin\left(\frac{\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right)+\cos\left(\frac{\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right)+\frac{2}{\Psi}\int_0^\tau\sin\left(\frac{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}\right)W(x)\frac{dx}{x^{2-\alpha-\beta}},$$
(21)

Solution:

Appling MK Elzaki transform Eq.(21)

$$E_{MK}\left[W(\tau)\right](\rho)=\frac{\rho^3}{1+\rho^2}+\frac{\rho^2}{1+\rho^2}+\frac{2\rho^2}{1+\rho^2}.E_{MK}\left[W(\tau)\right](\rho)$$

$$E_{MK}\left[W(\tau)\right](\rho)\left[\frac{1-\rho^2}{1+\rho^2}\right]=\frac{\rho^3+\rho^2}{1+\rho^2}$$

$$E_{MK}\left[W(\tau)\right](\rho)=\frac{\rho^2}{1-\rho}$$

Taking inverse MK Elzaki transform, we obtain

$$W(\tau)=e^{\frac{\tau^{\alpha+\beta-1}}{\Psi(\alpha+\beta-1)}}.$$

5. CONCLUSIONS

This work defined the Martinez–Kaabar fractal–fractional (MK FrFr) MK Elzaki transform, which was applied. Theorems and properties are crucial to this relatively new transformation, which can be found in the second-kind MK FrFr Volterra Integral Equation. The study of the MK Elzaki transform was successful in finding solutions.

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