Original Article
Bulletin of Pure and Applied Sciences.
Mathematics & Statistics Vol.44 E, No.2.
July-December 2025: P.70-81

Print version ISSN 0970 6577
Online version ISSN 2320 3226
Available online at www.bpasjournals.com

Generalized Record Values from the Power-Exponential Hazard Rate distribution and Characterizations

[1]Zaki Anwar, [2]Abdul Nasir Khan* and [3]Rafiqullah Khan

Author Affiliation:
[1,2,3]Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202002, India
[2]Department of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Pune-411038, Maharashtra

***Corresponding Author: Abdul Nasir Khan**, Department of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Pune-411038, Maharashtra.
E-mail: nasirgd4931@gmail.com

ABSTRACT
In this study we derive recurrence relations for the single and product moments of generalized record values (GRV's) based on the Power-Exponential hazard rate distribution. Additionally, these relations are formulated for the moments of upper record value (RV's). Furthermore, the study characterizes this distribution by utilizing recurrence relations for single and product moments, conditional expectation, and truncated moments.

Keyword: Order statistics, generalized record values, Power-exponential hazard rate distribution, single moments, product moments, recurrence relations and characterization.
AMS 2020 Subject Classifications: 62G30, 62E10, 60E05.

How to cite this article: Anwar Z., Khan A.N. and h Khan R. (2025). Generalized Record Values from the Power-Exponential Hazard Rate distribution and Characterizations. *Bulletin of Pure and Applied Sciences-Math & Stat.*, 44E (2), 70-81.

Received on 22.06.2025, Revised on 03.07.2025, Accepted on 20.08.2025

1. INTRODUCTION

The $RV's$ hold significant importance in everyday life. People are often interested in various records, including those related to sports, weather, crime statistics, economic trends and so on. Almost every measurable phenomenon is systematically recorded. The concept of $RV's$ was initially introduced by Chandler (1952) as a model for analyzing successive extreme values within a sequence of independent and identically distributed random variables. Later, Dziubdziela and Kopociński (1976) expanded this concept by applying it to a broader class of random variables, referring to them as k-th $RV's$. Subsequently, Minimol and Thomas (2013) termed these as GRV's, as the k −th term in the sequence of ordinary $RV's$ is also recognized as the k −th record value. By setting $k = 1$, one can obtain ordinary $RV's$.

Let $\{X_n, n \geq 1\}$ be a sequence of independently identically distributed (iid) continuous random variables with distribution function (df) $F(x)$ and probability density function (pdf). For fixed integer $k \geq 1$ we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of $k-$ th upper record times of $\{X_n, n \geq 1\}$ as follows:

$U_n^{(k)} = 1$

$U_{n+1}^{(k)} = \min\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1} \}$

where $X_{j:n}$ denotes the $j-$th order statistics in a sample of size n.

For $k=1$ and $n=1,2,...$ we write $U_n^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of $k-$th upper $RV's$ of $\{X_n, n \geq 1\}$. Note that for $k=1$ we have $Y_n^{(1)} = X_{U_n}, n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ (Ahsanullah (1995)). Moreover, we see that $Y_1^{(k)} = \min\{X_1, X_2, ..., X_n\} = X_{1:k}$. Then the pdf of $Y_n^{(k)}$ and joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are as follows:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^{k-1} f(x), \quad n \geq 1 \tag{1.1}$$

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln \overline{F}(x)]^{m-1} \frac{f(x)}{\overline{F}(x)}[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}$$
$$\times [\overline{F}(y)]^{k-1} f(y), \quad x < y, \quad 1 \leq m < n, \quad n \geq 2, \tag{1.2}$$

where $\overline{F}(x) = 1 - F(x)$.

The various developments on record values and related topics are extensively studied in the literature see; Balakrishnan and Ahsanullah (1995), Pawlas and Szynal (1998), Sultan (2007), Kumar and Khan (2012), Minimol and Thomas (2013), Khan and Khan (2019), MirMostafaee *et al.* (2016), Athar and Fawzy (2023).

Various probability distributions, including the Weibull, Rayleigh, exponential, and Gompertz, are extensively utilized for analyzing lifetime data. While these distributions exhibit monotonic hazard rate functions, they fail to adequately model bathtub-shaped hazard rate functions, which are frequently encountered in reliability and biological studies. To address this limitation, researchers have introduced extended versions of lifetime distributions that effectively capture bathtub-shaped hazard rates. These modified distributions offer greater flexibility and improved fitting capabilities. In this context, Tarvirdizade and Nematollahi (2017) proposed the Power-Exponential hazard rate distribution, which integrates the power hazard rate function with the exponential hazard rate function, thereby preserving the bathtub-shaped property. A random variable X follows Power-exponential hazard rate distribution (Tarvirdizade and Nematollahi (2017)), if it's pdf is of the form

$$f(x) = (\alpha x^{\beta} + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}, \quad x \geq 0, \quad \alpha, \ \beta, \ \gamma, \ \lambda > 0 \tag{1.3}$$

with df

$$F(x) = 1 - e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}, \quad x \geq 0, \quad \alpha, \ \beta, \ \gamma, \ \lambda > 0. \tag{1.4}$$

From (1.3) and (1.4), we have obtained

$$f(x) = (\alpha x^{\beta} + \lambda - \frac{\alpha \gamma}{1+\beta} x^{\beta+1} + \gamma[-\ln \overline{F}(x)]) \overline{F}(x). \tag{1.5}$$

The relation (1.5) will be used to establish some simple recurrence relations for moments of generalized upper record values from the Power-exponential hazard rate distribution.

Some sub-models of power-exponential hazard rate distribution are as follows:

Table 1: Sub-Models of Power-Exponential Hazard Rate Distribution

Parameters				Distribution
α	β	γ	λ	
-	0	-	0	exponential
-	1	-	0	Rayleigh
-	1	0		Linear Hazard Rate
-	-	-	0	power hazard rate
μ/θ	$\mu-1$	-	0	Weibull
$\mu\theta$	$\mu-1$	-	0	modified Weibull
0	-	-	-	Gompertz
-	0	-	-	Gompertz-Makeham

2. RELATIONS FOR SINGLE MOMENTS

Theorem 2.1. For the distribution given in (1.4) and $1 \le k \le n$, $j = 0,1,2,...$

$$\frac{\alpha\gamma k}{(\beta+1)(j+\beta+2)}\{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\}$$
$$= \frac{\alpha k}{(j+\beta+1)}\{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\} + \frac{\lambda k}{(j+1)}\{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\}$$
$$+ \frac{n\gamma}{(j+1)}\{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\} - E(Y_n^{(k)})^j . \tag{2.1}$$

Proof. From (1.1), we have

$$E(Y_n^{(k)})^j = \frac{k^n}{\Gamma(n)}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)dx . \tag{2.2}$$

From (1.5) and (2.2), we have

$$E(Y_n^{(k)})^j = \frac{\alpha k^n}{\Gamma(n)}\int_0^\infty x^{j+\beta}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx$$
$$+ \frac{\lambda k^n}{\Gamma(n)}\int_0^\infty x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx - \frac{\alpha\gamma k^n}{(1+\beta)\Gamma(n)}\int_0^\infty x^{j+\beta+1}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx$$
$$+ \frac{\gamma k^n}{\Gamma(n)}\int_0^\infty x^j[-\ln \bar{F}(x)]^n[\bar{F}(x)]^k dx . \tag{2.3}$$

On Integrating (2.3) by parts, treating x^j's for integration and rest of the integrand for differentiation and simplifying, we get

$$E(Y_n^{(k)})^j = \frac{\alpha k}{(j+\beta+1)}\{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\}$$
$$+ \frac{\lambda k}{(j+1)}\{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\} - \frac{\alpha\gamma k}{(\beta+1)(j+\beta+2)}\{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\}$$
$$+ \frac{n\gamma}{(j+1)}\{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\} .$$

On rearranging the terms of the above expression, we get the required expression.

Remark

i) For $k=1$ (2.1), the recurrence relation for single moments of upper record values from the Power-exponential hazard rate distribution has the form

$$\frac{\alpha\gamma k}{(\beta+1)(j+\beta+2)}\{E(X_{U(n)})^{j+\beta+2}-E(X_{U(n-1)})^{j+\beta+2}\}$$
$$=\frac{\alpha k}{(j+\beta+1)}\{E(X_{U(n)})^{j+\beta+1}-E(X_{U(n-1)})^{j+\beta+1}\}+\frac{\lambda k}{(j+1)}\{E(X_{U(n)})^{j+1}-E(X_{U(n-1)})^{j+1}\}$$
$$+\frac{n\gamma}{(j+1)}\{E(X_{U(n+1)})^{j+1}-E(X_{U(n)})^{j+1}\}-E(X_{U(n)})^{j}.$$

ii) Setting $\beta=\lambda=\gamma=0$ in (2.1), the result for single moments of generalized record values is deduced for exponential distribution as obtained by Balakrishnan and Ahsanullah (1995).

iii) Setting $\lambda=\gamma=0$, $\beta=1$ in (2.1), we deduced the recurrence relation for single moments of generalized record values from Rayleigh distribution as established by Khan *et al*. (2015).

iv) Setting $\lambda=\gamma=0$ and replace β, α by $\mu-1$ and $\mu\theta$ respectively in (2.1), we deduced the result for moments of generalized record values from Weibull distribution with shape μ and scale θ parameters as obtained by Pawlas and Szynal (2000).

v) Putting $\lambda=\gamma=0$ in (2.1), we get the recurrence relation for single moments of power hazard rate distribution as established by khan and khan (2019).

vi) Putting $\alpha=0$ in (2.1), the result for single moments of generalized record values is deduced for Gompertz distribution as established by Minimol and Thomas (2014).

vii) Setting $\beta=0$ in (2.1), we obtained the recurrence relation for moments of generalized record values from Gompertz-Makeham distribution.

3. RELATIONS FOR PRODUCT MOMENTS

The joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ is given by

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y)=\frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}f(y),\quad x>y,\quad 1\le m<n,\quad n\ge 2. \tag{3.1}$$

Theorem 3.1. For the distribution given in (1.2) and $k\ge 1$, $1\le m<n$, $n\ge 2$, $i,j=0,1,2,...$

$$\frac{\alpha\gamma k}{(\beta+1)(i+\beta+2)}\{E[(Y_m^{(k)})^{i+\beta+2}(Y_{n-1}^{(k)})^j]-E[(Y_{m-1}^{(k)})^{i+\beta+2}(Y_{n-1}^{(k)})^j]$$
$$=\frac{\alpha k}{(i+\beta+1)}\{E[(Y_m^{(k)})^{i+\beta+1}(Y_{n-1}^{(k)})^j]-E[(Y_{m-1}^{(k)})^{i+\beta+1}(Y_{n-1}^{(k)})^j]\}.$$
$$+\frac{\lambda k}{(i+1)}\{E[(Y_m^{(k)})^{i+1}(Y_{n-1}^{(k)})^j]-E[(Y_{m-1}^{(k)})^{i+1}(Y_{n-1}^{(k)})^j]\}$$
$$+\frac{\gamma m}{(i+1)}\{E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^j]-E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^j]\}-E[(Y_m^{(k)})^i(Y_n^{(k)})^j]. \tag{3.2}$$

Proof. From (3.1), we have

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j]=\frac{k^n}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^y x^i y^j[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dxdy. \tag{3.3}$$

From (1.5) and (3.3), we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{\alpha\, k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^{i+\beta} y^j [-\ln \overline{F}(x)]^{m-1} [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}$$

$$\times [\overline{F}(y)]^{k-1} f(y) dx\, dy + \frac{\lambda\, k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_y^\infty x^i y^j [-\ln \overline{F}(x)]^{m-1} [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}$$

$$\times [\overline{F}(y)]^{k-1} f(y) dx\, dy - \frac{\alpha \gamma k^n}{(\beta+1)\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_y^\infty x^{i+\beta+1} y^j [-\ln \overline{F}(x)]^{m-1}$$

$$\times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1} [\overline{F}(y)]^k f(y) dx\, dy + \frac{\gamma k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \overline{F}(x)]^m$$

$$\times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1} [F(y)]^{k-1} f(y) dx\, dy. \quad (3.4)$$

On Integrating (3.4) by parts, treating x^i 's for integration and rest of the integrand for differentiation and simplifying, we get

$$E(Y_n^{(k)})^j = \frac{\alpha\, k}{(i+\beta+1)} \{E[(Y_m^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j]\}$$

$$+ \frac{\lambda\, k}{(i+1)} \{E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j]\}$$

$$- \frac{\alpha \gamma k}{(\beta+1)(i+\beta+2)} \{E[(Y_m^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j]$$

$$+ \frac{\gamma m}{(i+1)} \{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]$$

On rearranging the terms of the above expression we get the required expression.

Remark

(i) For $k=1$ in (3.2), the recurrence relation for product moments of upper record values from the Power-exponential hazard rate distribution has the form

$$\frac{\alpha \gamma k}{(\beta+1)(i+\beta+2)} \{E[(X_{U(m)})^{i+\beta+2} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+\beta+2} (X_{U(n-1)})^j]$$

$$= \frac{\alpha\, k}{(i+\beta+1)} \{E[(X_{U(m)})^{i+\beta+1} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+\beta+1} (X_{U(n-1)})^j]$$

$$+ \frac{\lambda k}{(i+1)} \{E[(X_{U(m)})^{i+1} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+1} (X_{U(n-1)})^j]$$

$$+ \frac{\gamma m}{(i+1)} \{E[(X_{U(m+1)})^{i+1} (X_{U(n)})^j] - E[(X_{U(m)})^{i+1} (X_{U(n)})^j]\} - E[(X_{U(m)})^i (X_{U(n)})^j\}.$$

(ii) Setting $\beta = \lambda = \gamma = 0$ in (3.2), the result for product moments of generalized record values is deduced for exponential distribution.

(iii) Setting $\lambda = \gamma = 0$, $\beta = 1$ in (3.2), we deduced the recurrence relation for product moments of generalized record values from Rayleigh distribution.

(iv) Putting $\lambda = \gamma = 0$ in (3.2), we obtained the recurrence relation for product moments of power hazard rate distribution.

(v) Putting $\alpha = 0$ in (3.2), the result for product moments of generalized record values is deduced for Gompertz distribution as established by Minimol and Thomas (2014).

(vi) Setting $\beta = 0$ in (3.2), we obtained the recurrence relation for product moments of generalized record values from Gompertz-Makeham distribution.

(vii) Setting $\lambda = \gamma = 0$ and replace β, α by $\mu - 1$ and $\mu\theta$ respectively in (3.2), we deduced the result for product moments of generalized record values from Weibull distribution with shape μ and scale θ parameters.

Table 2: Moments of record values

n	$\alpha=1, \beta=1.5, \gamma=2$				$\alpha=2, \beta=1.5, \gamma=2$			
	$\lambda=2$				$\lambda=2$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.29201	0.12679	0.067263	0.04064	0.28636	0.12128	0.06265	0.036847
2	0.48778	0.28310	0.18421	0.13041	0.47663	0.26970	0.17098	0.11794
3	0.63221	0.44136	0.33185	0.26468	0.61682	0.41981	0.30770	0.23930
4	0.74584	0.59353	0.49768	0.43596	0.72734	0.56442	0.46160	0.39454
5	0.83923	0.73757	0.67402	0.63710	0.81848	0.70173	0.62582	0.57755
n	$\alpha=2, \beta=2.5, \gamma=3.5$				$\alpha=3, \beta=2.5, \gamma=3.5$			
	$\lambda=3$				$\lambda=3$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.18828	0.05204	0.01743	0.00661	0.18801	0.05185	0.01732	0.00655
2	0.31142	0.11433	0.04675	0.02069	0.31079	0.11381	0.04640	0.02048
3	0.40047	0.17587	0.08274	0.04112	0.39950	0.17496	0.08207	0.04066
4	0.46944	0.23388	0.12225	0.06651	0.46817	0.23256	0.12118	0.06573
5	0.52544	0.28789	0.16344	0.09568	0.52391	0.28618	0.16196	0.09451

4. CHARACTERIZATION

Theorem 4.1. Let k and j are integers such that $k \geq 1,\ j \geq 0$. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.4) is that

$$\frac{\alpha\gamma k}{(\beta+1)(j+\beta+2)}\{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\}$$
$$= \frac{\alpha k}{(j+\beta+1)}\{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\} + \frac{\lambda k}{(j+1)}\{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\}$$
$$+ \frac{n\gamma}{(j+1)}\{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\} - E(Y_n^{(k)})^{j}. \tag{4.1}$$

Proof. The necessary part follows from Theorem 4.1. On the other hand if the recurrence relation (4.1) is satisfied, then on using Khan *et al.* (2017), we have

$$\frac{n\gamma k^n}{(\beta+1)\Gamma(n)}\int_0^\infty x^{j+\beta+1}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^k\,dx = \frac{\lambda k^n}{\Gamma(n)}\int_0^\infty x^{j}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^k\,dx$$
$$+ \frac{\alpha k^n}{\Gamma(n)}\int_0^\infty x^{j+\beta}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^k\,dx - \frac{k^n}{\Gamma(n)}\int_0^\infty x^{j}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^{k-1}f(x)dx$$
$$+ \frac{\gamma k^{n+1}}{(j+1)\Gamma(n)}\int_0^\infty x^{j+1}[-\ln \overline{F}(x)]^{n-1}[\overline{F}(x)]^{k-1}f(x)\left\{-\ln \overline{F}(x) - \frac{n}{k}\right\}dx. \tag{4.2}$$

let

$$h(x) = -\frac{1}{k}[-\ln \bar{F}(x)]^n [\bar{F}(x)]^k \tag{4.3}$$

Differentiating both the sides of (4.3), we get

$$h'(x) = [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) \left\{ -\ln \bar{F}(x) - \frac{n}{k} \right\}$$

$$\frac{n\gamma k^n}{(\beta+1)\Gamma(n)} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \, dx = \frac{\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \, dx$$

$$+ \frac{\alpha k^n}{\Gamma(n)} \int_0^\infty x^{j+\beta} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \, dx - \frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx$$

$$+ \frac{\gamma k^{n+1}}{(j+1)\Gamma(n)} \int_0^\infty x^{j+1} h'(x) dx \,. \tag{4.4}$$

Integrating by parts last terms of (4.4) in the right hand side and using (4.3), we find that

$$\frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \left\{ \frac{f(x)}{\bar{F}(x)} - \alpha x^\beta - \lambda + \frac{\alpha\gamma}{(\beta+1)} x^{\beta+1} - \gamma[-\ln \bar{F}(x)] \right\} dx = 0 \,. \tag{4.5}$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin, 1984) to (4.5), we get

$$\frac{f(x)}{\bar{F}(x)} = \alpha x^\beta + \lambda - \frac{\alpha\gamma}{(\beta+1)} x^{\beta+1} + \gamma[-\ln \bar{F}(x)]$$

Which is the characterizing equation for the pdf as given in (1.3).

Hence the sufficient part proved.

Theorem 4.2. For fix a positive integer $k \geq 1$ and i, j are non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.3) is that

$$\frac{\alpha\gamma k}{(\beta+1)(i+\beta+2)} \{E[(Y_m^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j]$$

$$= \frac{\alpha k}{(i+\beta+1)} \{E[(Y_m^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j]\} \,.$$

$$+ \frac{\lambda k}{(i+1)} \{E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j]\}$$

$$+ \frac{\gamma m}{(i+1)} \{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]\} - E[(Y_m^{(k)})^i (Y_n^{(k)})^j] \,. \tag{4.6}$$

Proof. The necessary part follows from Theorem 2.1. On the other hand if the relation in (4.6) is satisfied, then to prove the sufficient part, we have consider

$$E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]$$

$$= -\frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^{i+1} y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2}$$

$$\times [\bar{F}(y)]^{k-1} f(y) \left\{ \ln \bar{F}(x) - \ln \bar{F}(y) - \frac{(n-m-1)[-\ln \bar{F}(x)]}{m} \right\} dx\, dy \,. \tag{4.7}$$

let

$$h(x,y)=\frac{1}{m}[-\ln\bar{F}(x)]^{m}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1} \tag{4.8}$$

Differentiating both the side of (4.8) with respect to x, we get

$$\frac{\partial}{\partial x}h(x,y)=[-\ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-2}$$
$$\times\left\{\ln\bar{F}(x)-\ln\bar{F}(y)-\frac{(n-m-1)[-\ln\bar{F}(x)]}{m}\right\} \tag{4.9}$$

From (4.7) and (4.9), we have

$$E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^{j}]-E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^{j}]$$
$$=\frac{k^{n}}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\left\{\int_0^y x^{i+1}\frac{\partial}{\partial x}h(x,y)dx\right\}y^{j}[\bar{F}(y)]^{k-1}f(y)dy\,. \tag{4.10}$$

Integrating (4.10) with respect to x and using (4.8), we get

$$E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^{j}]-E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^{j}]$$
$$=\frac{(i+1)k^{n}}{\Gamma(m+1)\Gamma(n-m)}\int_0^\infty\int_0^y x^{i}y^{j}[-\ln\bar{F}(x)]^{m}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}f(y)dxdy\,. \tag{4.11}$$

and

$$E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^{j}]-E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^{j}]$$
$$=-\frac{k^{n-1}}{\Gamma(m-1)\Gamma(n-m)}\int_0^\infty\int_0^y x^{i+1}y^{j}[-\ln\bar{F}(x)]^{m-2}\frac{f(x)}{\bar{F}(x)}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-2}$$
$$[\bar{F}(y)]^{k-1}f(y)\left\{\ln\bar{F}(x)-\ln\bar{F}(y)-\frac{(n-m-1)[-\ln\bar{F}(x)]}{m-1}\right\}dx\,dy\,. \tag{4.12}$$

let

$$h(x,y)=\frac{1}{(m-1)}[-\ln\bar{F}(x)]^{m-1}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}\,. \tag{4.13}$$

Differentiating both the side of (4.13) with respect to x, we get

$$\frac{\partial}{\partial x}h(x,y)=[-\ln\bar{F}(x)]^{m-2}\frac{f(x)}{\bar{F}(x)}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-2}$$
$$\times\left\{\ln\bar{F}(x)-\ln\bar{F}(y)-\frac{(n-m-1)[-\ln\bar{F}(x)]}{m-1}\right\}. \tag{4.14}$$

From (4.12) and (4.14), we have

$$E[(Y_m^{(k)})^{i+1}(Y_{n-1}^{(k)})^{j}]-E[(Y_{m-1}^{(k)})^{i+1}(Y_{n-1}^{(k)})^{j}]$$
$$=\frac{k^{n-1}}{\Gamma(m-1)\Gamma(n-m)}\int_0^\infty\left\{\int_0^y x^{i+1}\frac{\partial}{\partial x}h(x,y)dx\right\}y^{j}[\bar{F}(y)]^{k-1}f(y)dy\,. \tag{4.15}$$

Integrating (4.15) with respect to x and using (4.14), we get

$$E[(Y_m^{(k)})^{i+1}(Y_{n-1}^{(k)})^{j}]-E[(Y_{m-1}^{(k)})^{i+1}(Y_{n-1}^{(k)})^{j}]$$
$$=\frac{(i+1)k^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^y x^{i}y^{j}[-\ln\bar{F}(x)]^{m-1}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}f(y)dxdy\,. \tag{4.16}$$

Using (4.11) and (4.16), (4.6) can be written as

$$\frac{k^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^y x^i y^j[-\ln\overline{F}(x)]^{m-1}[\ln\overline{F}(x)-\ln\overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)$$
$$\times\left\{-\frac{f(x)}{\overline{F}(x)}+\alpha x^{\beta}+\lambda-\frac{\alpha\gamma x^{\beta+1}}{(\beta+1)}+\gamma[-\ln\overline{F}(x)]\right\}dx\,dy\,. \quad (4.17)$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin, 1984) to (4.17), we get

$$\frac{f(x)}{\overline{F}(x)}=\alpha x^{\beta}+\lambda-\frac{\alpha\gamma}{(\beta+1)}x^{\beta+1}+\gamma[-\ln\overline{F}(x)]$$

Which is the characterizing equation for the pdf as given in (1.3).
Hence the sufficient part proved.

Theorem 4.3. Let X be an absolutely continuous non-negative random variable having $df\ \ F(x)$, with $F(0)=0$ and $0\le F(x)\le 1\ \forall\ 0<x<\infty$, then

$$E\left(\left\{\frac{\alpha}{\beta+1}(Y_n^{(k)})^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma Y_n^{(k)}}-1)\right\}|\,(Y_m^{(k)})=x\right)=\frac{\alpha}{\beta+1}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)+\frac{n-m}{k}\,. \quad (4.18)$$

if and only if

$$F(x)=1-e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)},\quad x\ge 0,\ \ \alpha,\ \beta,\gamma,\ \ \lambda>0\,.$$

Proof. From (1.1) and (1.2), we have

$$E\left(\left\{\frac{\alpha}{\beta+1}(Y_n^{(k)})^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma Y_n^{(k)}}-1)\right\}|\,(Y_m^{(k)})=x\right)$$
$$=\frac{k^{n-m}}{\Gamma(n-m)}\int_x^\infty\left\{\frac{\alpha}{(\beta+1)}y^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma y}-1)\right\}\left(-\ln\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{n-m-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{k-1}\frac{f(y)}{\overline{F}(x)}dy\,. \quad (4.19)$$

By setting $z=\dfrac{\overline{F}(y)}{\overline{F}(x)}=\dfrac{e^{-\left(\frac{\alpha}{\beta+1}y^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma y}-1)\right)}}{e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}}$ in (4.19), we have

$$E\left(\left\{\frac{\alpha}{\beta+1}(Y_n^{(k)})^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma Y_n^{(k)}}-1)\right\}|\,(Y_m^{(k)})=x\right)$$
$$=\frac{k^{n-m}}{\Gamma(n-m)}\left(\left\{\frac{\alpha}{(\beta+1)}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)\right\}\int_0^1(-\ln z)^{n-m-1}z^{k-1}dz-\int_0^1(-\ln z)^{n-m}z^{k-1}dz\right). \quad (4.20)$$

In view of Gradshteyn and Ryzhik (2007, p-551), note that

$$\int_0^1(-\ln x)^{\mu-1}x^{\upsilon-1}dx=\frac{\Gamma\mu}{\upsilon^{\mu}},\quad \mu>0\,,\ \upsilon>0\,. \quad (4.21)$$

Using (4.21) in (4.20), we obtain the result given in (4.18).
To prove the sufficient part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)}\int_x^\infty \left\{\frac{\alpha}{\beta+1}y^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma y}-1)\right\}[\ln \overline{F}(x)-\ln \overline{F}(y)]^{n-m-1}$$
$$\times[\overline{F}(y)]^{k-1}f(y)dy=[\overline{F}(x)]^k g_{n|m}(x), \quad (4.22)$$

where

$$g_{s|r}(x)=\frac{\alpha}{\beta+1}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)+\frac{n-m}{k}.$$

Differentiating both sides of (4.22) with respect to x, we get

$$-\frac{k^{n-m}f(x)}{\Gamma(n-m-1)\overline{F}(x)}\int_x^\infty \left\{\frac{\alpha}{\beta+1}y^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma y}-1)\right\}[\ln \overline{F}(x)-\ln \overline{F}(y)]^{n-m-2}$$
$$\times[\overline{F}(y)]^{k-1}f(y)dy=g'_{n|m}(x)[\overline{F}(x)]^k+k\,g_{n|m}(x)[\overline{F}(x)]^{k-1}f(x)$$

or

$$-k\,g_{n|m+1}(x)[\overline{F}(x)]^{k-1}f(x)=g'_{n|m}(x)[\overline{F}(x)]^k-k\,g_{n|m}(x)[\overline{F}(x)]^{k-1}f(x).$$

Therefore

$$\frac{f(x)}{\overline{F}(x)}=-\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x)-g_{n|m}(x)]},$$
$$=\alpha x^\beta+\lambda e^{\gamma x}, \quad (4.23)$$

where

$$g'_{n|m}(x)=\alpha x^\beta+\lambda e^{\gamma x}$$
$$g_{n|m+1}(x)-g_{n|m}(x)=-\frac{1}{k}.$$

Integrating both the sides of (4.23) with respect to x between $(0,y)$, Hence the sufficiency part has proved.

Theorem 4.4. Suppose that X be an absolutely continuous (with respect to Lebesque measure) random variable with the *df* $F(x)$ and *pdf* $f(x)$ $\forall\ 0<x<\infty$, such that $f'(x)$ and $E(X\mid X\le x)$, exist for all x, $0<x<\infty$, then

$$E(X\mid X\le x)=g(x)\eta(x), \quad (4.24)$$

where

$$\eta(x)=\frac{f(x)}{F(x)}$$

and

$$g(x)=-\frac{x}{\alpha x^\beta+\lambda e^{\gamma x}}+\frac{e^{\left(\frac{\alpha}{(\beta+1)}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}}{\alpha x^\beta+\lambda e^{\gamma x}}\int_0^x e^{\left(\frac{\alpha}{(\beta+1)}u^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma u}-1)\right)}du$$

if and only if

$$f(x)=(\alpha x^\beta+\lambda e^{\gamma x})e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)},\quad x\ge 0,\ \ \alpha,\ \beta,\ \gamma\ \ \lambda>0.$$

Proof. From (1.3), we have

$$E(X\mid X\le x)=\frac{1}{F(x)}\int_0^x u(\alpha u^\beta+\lambda e^{\gamma u})e^{-\left(\frac{\alpha}{\beta+1}u^{\beta+1}+\frac{\lambda}{\gamma}(e^{\gamma u}-1)\right)}du. \quad (4.25)$$

Integrating (4.25) by parts, treating $(\alpha x^{\beta} + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x} - 1)\right)}$ for integration and rest of the integrand for differentiation, we get

$$E(X \mid X \le x) = \frac{1}{F(x)} \left\{ - x e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x} - 1)\right)} + \int_0^x e^{-\left(\frac{\alpha}{\beta+1} u^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma u} - 1)\right)} du \right\}. \quad (2.26)$$

Now dividing and multiplying (4.27) by $f(x)$, we obtain the result as given in (4.24).

For proving sufficient part, we have from (4.24)

$$\int_0^x u\, f(u) du = g(x) f(x). \quad (4.27)$$

Differentiating (4.27) on both sides with respect to x, we find that

$$x f(x) = g'(x) f(x) + g(x) f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}$$ [Ahsanuallah, *et. al* (2016)]

$$\frac{f'(x)}{f(x)} = -(\alpha x^{\beta} + \lambda e^{\gamma x}) + \frac{\alpha\beta x^{\beta-1} + \lambda \gamma e^{\gamma x}}{\alpha x^{\beta} + \lambda e^{\gamma x}}, \quad (4.28)$$

where

$$g'(x) = x - g(x)\left(-(\alpha x^{\beta} + \lambda e^{\gamma x}) + \frac{\alpha\beta x^{\beta-1} + \lambda \gamma e^{\gamma x}}{\alpha x^{\beta} + \lambda e^{\gamma x}} \right).$$

On integrating (4.28) both sides with respect to x, we get

$$f(x) = C(\alpha x^{\beta} + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} e^{\gamma x}\right)}.$$

Further, To obtain the value of C (constant of integration), we have used the property of pdf, note that

$$\int_0^{\infty} f(x) dx = 1.$$

Thus

$$\frac{1}{C} = \int_0^{\infty} (\alpha x^{\beta} + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} e^{\gamma x}\right)} dx = \frac{1}{e^{\lambda/\gamma}},$$

which proves that

$$f(x) = (\alpha x^{\beta} + \lambda e^{\gamma x}) e^{\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x} - 1)\right)}, \quad x > 0, \quad \alpha, \ \beta, \ \lambda > 0.$$

REFERENCES

1. Ahsanullah, M. (1995): *Record Statistics*. Nova Science Publishers, New York.
2. Ahsanullah, M., Shakil, M. and Golam Kibria, B.M (2016): Characterization of continuous distribution by truncated moment. *Journal of Modern Applied Statistical Method*, 15, 316-331.
3. Athar, H., and Fawzy, M. A. (2023): Moment properties of record values from rayleigh lomax distribution and characterization. Journal of Modern Applied Statistical Methods, 22(2).
4. Balakrishnan, N. and Ahsanullah, M. (1995): Relations for single and product moments of record values from exponential distribution, *J. Appl. Statist. Sci.*, 2, 73-87.

5. Chandler, K.N. (1952): The distribution and frequency of record values. *J. Roy. Statist. Soc. Ser. B*, 14, 220-228.
6. Dziubdziela, W. and Kopociński, B. (1976): Limiting properties of the $k-$th record value, *Appl. Math.*, 15, 187-190.
7. Gradshteyn, I.S. and Ryzhik, I.M. (2007): *Tables of Integrals, Series of Products.* Academic Press, New York.
8. Hwang, J.S. and Lin, G.D. (1984): On a generalized moment's problem II. *Proc. Amer. Math. Soc.*, 91, 577-580.
9. Khan, M.I. and Khan, M.A.R. (2019): Generalized record values from distributions having Power hazard function and characterization. *Journal of Statistics Applications and probability*, 8, 103-111.
10. Khan, R.U., Khan, M. A. and Khan, M.A. R (2017): Relations for moments of generalized record values from additive Weibull distribution and associated inference. *Statistics, Optimization and Information Computing*, 5, 127-136.
11. Khan, R.U., Kulshrestha, A. and Khan, M.A. (2015): Relations for moments of $k-$th record values from exponential-Weibull lifetime distribution and a characterization. *J. Egyptian Math. Soc.*, 23, 558-562.
12. Kumar, D., and Khan, M. I. (2012): Recurrence relations for moments of k record values from generalized beta II distribution and a characterization, *Selcuk J. Appl. Math.*,13(1),75-82.
13. Minimol, S. and Thomas, P.Y. (2014): On characterization of Gompertz distribution by properties of generalized record values. *J. Stat. Theory Appl.*, 13, 38-45.
14. Minimol, S., and Thomas, P. Y. (2013): On some properties of Makeham distribution using generalized record values and its characterization, *Braz J Probab Stat*, 27(4):487–501
15. MirMostafaee, S. T., Asgharzadeh, A., and Fallah, A. (2016). Record values from NH distribution and associated inference. *Metron*, 74, 37-59.
16. Pawlas, P. and Szynal, D. (1998): Relations for single and product moments of $k-$th record values from exponential and Gumble distributions. *J. Appl. Statist. Sci.*, 7, 53-62.
17. Pawlas, P. and Szynal, D. (2000): Recurrence relations for single and product moments of *k*-th record values from Weibull distributions and a characterization, *J. Appl. Statist. Sci.*, 10, 17-26.
18. Sultan, K. S. (2007). Record values from the modified Weibull distribution and applications. *International Mathematical Forum* (Vol. 41, No. 2, pp. 2045-2054).
19. Tarvirdizade, B. and Nematollahi, N. (2017): A new flexible hazard rate distribution: application and estimation of its parameters. *Comm. Statist. Simulation Comput.*, 48, 882-899.

****