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A Study On Partial Square Sum Processes

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Abstract
In this article we introduced partial square sum processes. Let $\{X_n, n = 1,2,..\}$ be a sequence of independent and non-negative random variables and let $G(x)$ be the distribution function of X. Then $\{X_n, n = 1,2, ...\}$ is called a partial square sum process, if the distribution function of X_{n+1} is $G(\beta_n^2 x)$ $(n = 1,2, ...)$ where $\beta_n^2 > 0$ are constants and $\beta_n^2 = \beta_1^2 + \beta_2^2 + \cdots + \beta_{n-1}^2$. We study some properties of partial square sum process.
Keywords: geometric process, partial product process, partial square sum process.
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1 Introduction

A stochastic process $\{X_n, n = 1,2, ...\}$ is said to be a Geometric process, if there exists a real $\beta_0 > 0$ such that $\{\beta_0^{n-1} X_n, n = 1,2, ...\}$ forms a renewal process. The positive number β_0 is called the ratio of the G.P. A Geometric process is stochastically increasing if the ratio $0 < \beta_0 \leq 1$. It is stochastically decreasing if the ratio $\beta_0 > 1$. A Geometric process will become a renewal process if the ratio $\beta_0 = 1$. Therefore Geometric process is a simple monotone process and is a generalization of the renewal process. Assume that $\{X_n, n = 1,2, ...\}$ is a Geometric process with ratio β_0. Let the Distribution function and density function of X_1 be G and g respectively and denote $E(X_1) = \lambda$ and $Var(X_1) = \sigma^2$. Then $E(X_n) = \frac{\lambda}{\beta_0^{2^{n-1}}}$ and $Var(X_n) = \frac{\sigma^2}{\beta_0^{2^n}}$. Thus β_0, λ and σ^2 are three important parameters of a Geometric process. Let $\{X_n, n = 1,2,..\}$ be a sequence of non-negative independent random variables and let $G(x)$ be the distribution function of X_1. Then $\{X_n, n = 1,2, ...\}$ is called a partial product process, if the distribution function of X_{i+1} is $G(\beta_k x)$ $(k = 1,2, ...)$ where $\beta_k > 0$ are constants and $\beta_k = \beta_0 \beta_1 \beta_2 ... \beta_{k-1}$.

2 Partial square sum process

Let $\{X_n, n = 1,2,..\}$ be a sequence of independent and non-negative random variables and let $G(x)$ be the distribution function of X. Then $\{X_n, n = 1,2, ...\}$ is called a partial square sum process, if the distribution function of X_{n+1} is $G(\beta_n^2 x)$ $(n = 1,2, ...)$ where $\beta_n^2 > 0$ are constants and $\beta_n^2 = \beta_1^2 + \beta_2^2 + \cdots + \beta_{n-1}^2$.

Lemma 2.1. $\beta_n^2 = \beta_0^2 + \beta_1^2 + \cdots + \beta_{n-1}^2$ show that $\beta_n^2 = 2^{n-1}\beta_0^2$

Proof. When $n = 1$, $\beta_1^2 = \beta_0^2$. Thus, the result is true for $n = 1$.
Assume that the result is true for $n = k$.

$$\beta_k^2 = 2^{k-1}\beta_0^2 \ldots (1)$$

Then we have to prove that the result is true for $n = k + 1$.

$$\beta_{k+1}^2 = \beta_0^2 + \beta_1^2 + \cdots + \beta_{k-1}^2 + \beta_k^2$$
$$= \beta_k^2 + \beta_k^2 \text{ [From(1)]}$$
$$= 2\,\beta_k^2$$
$$= 2 \times 2^{k-1}\beta_0^2$$
$$\beta_{k+1}^2 = 2^{k+1-1}\,\beta_0^2 = 2^k\beta_0^2$$

Thus, the result is true for $n = k + 1$ also.

$$\beta_n^2 = 2^{n-1}\beta_0^2;\ n = 1,2,3, \ldots$$

Remark 2.2. By Lemma 2.1, the distribution function of X_{k+1} is $G(2^{n-1}\beta_0^2 x)$ for $n = 1,2,\ldots$ the term β_0^2 as the ratio of the partial square sum process.

Lemma 2.3. The partial square sum process $\{X_n, n = 1,2,\ldots\}$ is

(i) Stochastically decreasing if $\beta_0^2 > 1$

(ii) Stochastically increasing if $0 < \beta_0^2 < 1$.

Proof. Let $\beta_0^2 > 1$. Note that for any $\gamma \geq 0$

$$G(\gamma) \leq G(2^2\beta_0\gamma)) \leq G((2^2)^2\beta_0\gamma) \leq \cdots \leq G((2^2)^{n-1}\beta_0\gamma)$$

This implies

$$P(X_1 \geq \gamma) \geq P(X_2 \geq \gamma) \geq P(X_3 \geq \gamma) \ldots \geq P(X_n \geq \gamma)$$

This implies that $\{ X_n, n = 1,2,\ldots\}$ is Stochastically decreasing if $\beta_0^2 > 1$ Similarly we can prove that it is Stochastically increasing if $0 < \beta_0^2 < 1$.

Lemma 2.4. Let $E(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2$. Then for $n = 1,2,\ldots$

$$E(X_{n+1}) = \frac{\mu}{2^{n-1}\beta_0^2}, \mathrm{Var}(X_{n+1}) = \frac{\sigma^2}{2^{2(n-1)}\beta_0^4}$$

Proof. By Lemma 2.3, for $n = 1,2,3,\ldots$ the density function of X_{n+1} is $2^{n-1}\beta_0^2 g(2^{n-1}\beta_0^2 x)$ where g is the density function of X_1.

Now $E(X_{n+1}) = \int x\,[2^{n-1}\,\beta_0^2 g(2^{n-1}\,\beta_0^2 x)]dx$

$$= 2^{n-1}\,\beta_0^2 \int x\,[g(2^{n-1}\,\beta_0^2 x)]dx$$

$$= 2^{n-1}\,\beta_0^2 \int \left(\frac{y}{2^{n-1}\,\beta_0^2}\right) g(y)\,\frac{dy}{2^{n-1}\,\beta_0^2} \qquad \text{where} \quad y = 2^{n-1}\,\beta_0^2$$

$$= \frac{1}{2^{n-1}\beta_0^2}\int y\,g(y)\,dy$$

$$= \frac{1}{2^{n-1}\beta_0^2}\,E(X_1)$$

Therefore $E(X_{n+1}) = \frac{\mu}{2^{n-1}\beta_0^2}$

Now $E(X_{n+1}^2) = \int x^2\,[2^{n-1}\,\beta_0^2\,g(2^{n-1}\,\beta_0^2 x)]dx$

$$= 2^{n-1}\,\beta_0^2 \int x^2\,g(2^{n-1}\,\beta_0^2 x)dx$$

$$= 2^{n-1}\,\beta_0^2 \int \left(\frac{y}{2^{n-1}\,\beta_0^2}\right)^2 g(y)\,\frac{dy}{2^{n-1}\,\beta_0^2} \qquad \text{where } y = 2^{n-1}\,\beta_0^2$$

$$= \frac{1}{(2^{n-1}\beta_0^2)^2}\int y^2\,g(y)\,dy$$

$$= \frac{1}{2^{2(n-1)}\beta_0^4}\,E(X_1^2)$$

$$= \frac{E(X_1^2)}{2^{2(n-1)}\beta_0^4}$$

$$Var(X_{n+1}) = E(X_{k+1}^2) - [E(X_{k+1})]^2$$

$$= \frac{E(X_1^2)}{2^{2(n-1)}\beta_0^4} - \left[\frac{E(X_1)}{2^{n-1}\beta_0^2}\right]^2 = \frac{E(X_1^2)}{2^{2(n-1)}\beta_0^4} - \frac{[E(X_1)]^2}{2^{2(n-1)}\beta_0^4}$$

$$= \frac{\sigma^2}{2^{2(n-1)}\,\beta_0^4}$$

3 Properties of Partial Square Sum Process

In this section we study some properties of partial square sum process. Let G and g be the distribution function and density function of X_1 respectively. and denote $E(X_1) = \mu$, and $\mathrm{Var}(X_1) = \sigma^2$. Then by Lemma 2.4, for $i = 1,2,\ldots$. We have

$$E(X_{i+1}) = \frac{\mu}{2^{i-1}\beta_0^2}$$

$$\text{and } \mathrm{Var}(X_{i+1}) = \frac{\sigma^2}{2^{2(i-1)}\beta_0^4}$$

Thus β_0^2, μ and σ^2 are important three parameters of partial square sum process.

Define $u_0 = 0$ and $u_n = \sum_{i=1}^{n} X_i$

Let $G_n = \sigma\,(X_1, X_2, \ldots, X_n)$ be the σ-algebra generated by $\{X_i, i = 1,2,\ldots n\}$.

Theorem 3.1. If $\beta_0^2 > 1$, the $\{U_n, n = 1,2, \ldots\}$ is a nonnegative submartingale with respect to $G_n = \sigma(X_1, X_2, \ldots, X_n)$.

Proof. Obviously $\{U_n, n = 1,2, \ldots\}$ is a sequence of increasing nonnegative random variables with

$$E[U_{n+1}|G_n] = U_n + E[X_{n+1}] \geq U_n \ldots (1)$$

Also $\sup_{n\geq 0} E[|U_n|] = \lim_{n\to\infty} E[U_n]$

$$= \lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} E[X_i]$$

$$= \lim_{n\to\infty}\left[\mu + \sum_{i=2}^{n} \frac{\mu}{2^{i-1}\beta_0^2}\right]$$

$$= \mu\left[1 + \sum_{i=2}^{n} \frac{1}{2^{i-1}\beta_0^2}\right] < \infty \ldots (2)$$

Where eqn (2) is due to the fact that the series $\sum_{i=2}^{n} \frac{1}{2^{i-1}\beta_0^2}$ is convergent by comparing it with geometric series if $\frac{1}{\beta_0^2} < 1$.

Thus from eqn (1) and (2), by definition $\{U_n, n = 1,2, \ldots\}$ is a nonnegative submartingale with respect to $\{G_n, n = 1,2, \ldots\}$ if $\beta_0^2 > 1$.

Theorem 3.2. If $\beta_0^2 > 1$, there exist a random variable U such that the sequence $\{U_n\}$ converges almost sure to U.

Proof. $\{U_n, n = 1,2, \ldots\}$ is a nonnegative sub martingale with respect to $\{G_n, n = 1,2, \ldots\}$

$$\sup_{1\leq n\leq\infty} E[|U_n|] \leq \infty$$

Then with probability 1, random variable $U = \lim_{n\to\infty} U_n$ and $E[|U|] < \infty$

∴ the sequence $\{U_n\}$ converges almost sure to U.

Theorem 3.3. If $\beta_0^2 > 1$, $\{U_n, n = 1,2, \ldots\}$ has a unique decomposition such that

$$U_n = M_n - B_n \ldots (3)$$

Where $\{M_n, n = 1,2, \ldots\}$ is a martingale. $\{B_n, n = 1,2, \ldots\}$ is a decreasing with $B_1 = 0$ and $B_n \in G_{n-1}$.

Proof. Let $M_1 = U_1$ and $B_1 = 0$ for $n \geq 2$

We define $M_n = M_{n-1} + (U_n - E[U_n \mid G_{n-1}]) \ldots (4)$

$$B_n = B_{n-1} + (U_{n-1} - E[U_n \mid G_{n-1}]) \ldots (5)$$

From eqn. (4) and (5) we have

$$M_n - B_n = \sum_{i=2}^{n} (U_i - U_{i-1}) + U_1 - B_1 = U_n$$

and (3) follows. It checks $\{M_n, n = 1,2, \ldots\}$ and $\{B_n, n = 1,2, \ldots\}$ satisfy the terms.

Next to prove that a decomposition is unique.

Suppose $U_n = M_n^* - B_n^*$ is another decomposition.

Then $M_n = M_n^* = B_n - B_n^*$

Since $M_2 = M_2^* = B_2 - B_2^* \in G_1$

implies that $M_2 = M_2^*$. Then by induction

Hence $B_n = B_n^*$.

Definition 3.4. Given a partial square sum process $\{X_j, j = 1,2,3, \ldots\}$ and if $w(t) = \sup_{j\in Z}\{j\colon V_j \leq 1\}$, where $V_j = \sum_{i=1}^{j} X_j$ the age at t is defined by $A(t) = t - U_{w(t)}$ the residual life at t is defined by

$$B(t) = U_{w(t)+1} - t$$

and the total life at t is defined by

$$X_{w(t)+1} = U_{w(t)+1} - U_{w(t)}$$
$$= A(t) + B(t)$$

Let G_j be the distribution function of U_j and $G_j(x) = 1 - U_j(x)$.

Theorem 3.5. If G is the distribution function of X_1 and $\bar{G}(x)=1-G(x)$.
Then

1. $P(A(t)>x)=\begin{cases}\bar{G}(t)+\sum_{n=1}^{\infty}\int_0^{t-x}\bar{G}(2^{n-1}\beta_0^2(t-u)dG_n(t), & 0<x<t\\ 0, & x\geq t\end{cases}$
2. $P(B(t)>x)=\begin{cases}\bar{G}(t+x)+\sum_{n=1}^{\infty}\int_0^{t}\bar{G}(2^{n-1}\beta_0^2(x+t-y)\,dG_n(y), & x>0\\ 1, & x\leq 0\end{cases}$
3. $P\big(X_{W(t)+1}>x\big)=\begin{cases}\bar{G}(t\vee x)+\sum_{n=1}^{\infty}\int_0^{t}\bar{G}(2^{n-1}\beta_0^2(x\vee t-y)dG_n(y), & x>0\\ 1, & x\leq 0\end{cases}$
4. $P\big(U_{W(t)}\leq x\big)=\begin{cases}\bar{G}(t)+\sum_{n=1}^{\infty}\int_0^{t-x}\bar{G}(2^{n-1}\beta_0^2(t-y)dG_n(y), & 0\leq x\leq t\\ 1, & x>t\end{cases}$

Proof. 1. Assume $0<x<t$

$$\begin{aligned}P(A(t)>x)&=P\big(U_{W(t)}<t-x\big)\\ &=\sum_{j=0}^{\infty}P\big(U_{W(t)}<t-x,W(t)=j\big)\\ &=\sum_{j=0}^{\infty}P\big(U_j<t-x,U_{W+1}>t\big)\\ &=\bar{G}(t)+\sum_{j=1}^{\infty}\int_0^{t-x}P\big(U_j>t\,\big|\,U_j=x)\,dG_j(x)\\ &=\bar{G}(t)+\sum_{j=1}^{\infty}\int_0^{t-x}P(X_{j+1}>t-x)\,dG_j(x)\\ &=\bar{G}(t)+\sum_{j=1}^{\infty}\int_0^{t-x}\bar{G}(2^{j-1}\beta_0^2(t-u)dG_j(u)\end{aligned}$$

For $x\geq t$, $P(A(t)>x)=0$ is trivial.
Part (i) proof is completed.

2. Assume $x>0$

$$\begin{aligned}P(B(t)>x)&=P\big(U_{W(t)}>t+x\big)\\ &=\sum_{j=0}^{\infty}P\big(U_{W(t)+1}>t+x,W(t)=j\big)\\ &=\sum_{j=0}^{\infty}P\big(U_{j+1}>t+x,U_j\leq t\big)\\ &=\bar{G}(t)+\sum_{j=1}^{\infty}\int_0^{t}P\big(U_{j\mp1}>t+x\,\big|\,U_j=y)\,dG_j(y)\\ &=\bar{G}(t+x)+\sum_{j=1}^{\infty}\int_0^{t}P(X_{j+1}>t+x-y)\,dG_j(y)\\ &=\bar{G}(t+x)+\sum_{j=1}^{\infty}\int_0^{t}\bar{G}(2^{j-1}\beta_0^2(x+t-y)dG_j(y)\end{aligned}$$

For $y\leq 0$, $P(B(t)>x)=1$ is trivial.
Part (2) proof completed.
3. Assume $x>0$

$$\begin{aligned}P\big(X_{W(t)+1}>x\big)&=\sum_{j=0}^{\infty}P\big(X_{W(t)+1}>x,W(t)=j\big)\\ &=\sum_{j=1}^{\infty}\int_0^{t}P\big(X_{j+1}>x,U_j\leq t\leq U_{j+1}\big|\,U_j=y)\,dG_j(y)\end{aligned}$$

$$= \sum_{j=1}^{\infty} \int_0^t P(X_{j+1} > max\,(x, t-y)\ dG_j(y) \ldots (6)$$

$$= \bar{G}(t \vee x) + \sum_{j=1}^{\infty} \int_0^t \bar{G}(2^{j-1}\beta_0^2(x \vee (t-y))dG_j(y)$$

For $x \le 0$, $P\left(X_{W(t)+1} > x\right) = 1$ is trivial.

Part (3) proof completed.

4. Assume $0 < x \le t$

$$\begin{aligned}
P\left(U_{W(t)+1} \le x\right) &= P\left(t - U_{W(t)} \le x\right) \\
&= P\left(U_{W(t)} \ge t - x\right) \\
&= \sum_{j=0}^{\infty} P\left(U_{W(t)} \ge t - x, W(t) = j\right) \\
&= \sum_{j=0}^{\infty} P\left(U_{j+1} > t - x, U_{j+1} > t\right) \\
&= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^t P\left(U_{j\mp 1} > t \,\middle|\, U_j = x\right) dG_j(x) \\
&= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-2} P(X_{j+1} > t - x)\, dG_j(x) \\
&= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} \bar{G}(2^{j-1}\beta_0^2(t-u))dG_j(u)
\end{aligned}$$

For $x \ge t$, $P\left(U_{W(t)} \le x\right) = 1$ is trivial.

Part (4) proof completed.

Limit Theorems for partial square sum process

Theorem 3.6. Weld's Equation for partial square sum process

Suppose $\{X_n, n = 1,2,3, \ldots\}$ forms a partial square sum process with ratio β_0^2 with $E(X_1) = \mu < \infty$ then for $t > 0$, we have

$$E\left(U_{W(t)+1}\right) = \mu \mathrm{E}\left[1 + \sum_{n=2}^{W(t)+1} \frac{1}{2^{n-2}\beta_0^2}\right]$$

Proof. Let I_A be the indicator function of event A. Then $I_{\{U_{n-1 \le t}\}} = I_{\{W_{n+1 \ge n}\}}$ and X_n are independent. Consequently, for $t > 0$ we get

$$\begin{aligned}
E\left(U_{W(t)+1}\right) &= \mathrm{E}\left[\sum_{n=1}^{W(t)+1} X_n\right] \\
&= \sum_{n=1}^{\infty} E\left[X_n I_{\{W_{n+1 \ge n}\}}\right] \\
&= \sum_{n=1}^{\infty} E(X_n) P(W_{(t)+1} \ge n) \\
&= \sum_{j=1}^{\infty} \left(\sum_{n=1}^{j} E(X_n)\right) P(W_{(t)+1} = j) \\
&= E\left[E(X_1) + \sum_{n=2}^{W(t)+1} E(X_n)\right] \\
&= \mu \mathrm{E}\left[1 + \sum_{n=2}^{W(t)+1} \frac{1}{2^{n-2}\beta_0^2}\right]
\end{aligned}$$

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