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Applications Of Fractional Chromatic Numbers In Mycielski's Graphs

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ABSTRACT

This chapter explores the applications of Fractional Chromatic Numbers and transformative properties of Mycielski's graph construction, shedding light on the relationships between fractional chromatic numbers, clique numbers, and chromatic numbers. The Mycielski transformation, applied iteratively to a starting graph, generates a sequence of graphs, each revealing intriguing patterns in these graph parameters. We prove that, for any graph G, the resulting graph $\mu(G)$ exhibits distinct properties:

(a) The clique number $\omega(\mu(G))$ remains equal to the original graph's $\omega(G)$ for all iterations.

(b) The chromatic number $\chi(\mu(G))$ increases by one compared to the chromatic number of the previous iteration, i.e., $\chi(\mu(G)) = \chi(G) + 1$.

(c) The fractional chromatic number $\chi_F(\mu(G))$ increases by a fraction dependent on the original graph's fractional chromatic number $\chi_F(G)$.

***Index Terms*—Graph coloring, fractional coloring, fractional chromatic number**

1. INTRODUCTION

The thesis explores the intricate realm of graph theory, specifically delving into the fascinating domain of fractional chromatic numbers and Mycielski graphs. Graph theory serves as a powerful tool for modeling and understanding complex relationships within diverse systems, making it an invaluable discipline with applications ranging from computer science to network optimization. The fractional chromatic number, a concept deeply rooted in graph coloring, offers insights into the chromatic properties of graphs with real-valued color assignments.

Complementing this exploration is an in-depth analysis of Mycielski graphs, constructed through a transformative process that unveils intriguing

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connections between graphs and their transformed counterparts. The interplay between fractional chromatic numbers and Mycielski graphs unfolds a rich tapestry of theoretical nuances and practical applications.

This chapter aims to contribute to the evolving landscape of graph theory by unraveling the complexities inherent in these two interconnected concepts, shedding light on their theoretical foundations, and exploring their potential applications across various domains. Through a meticulous examination of existing literature and the development of novel insights, this thesis aspires to deepen our understanding of the intricate structures and properties that underlie graph theory, ultimately contributing to the broader landscape of mathematical research and its practical implications.

The literature review encompasses a wide array of studies exploring the fractional chromatic number and related graph theory topics. Reed (2002) delves into the realm of large graph colorings, emphasizing algorithmic aspects within networking contexts. Mohar and Zhu (2004) contribute insights into fractional chromatic numbers and their connection to ordered sets, enhancing the combinatorial theory of graphs. Mahdian and Peeters (2003) investigate the fractional chromatic number of graphs, providing theoretical perspectives on coloring problems. Kratochvíl (1995) focuses on the fractional chromatic number of triangle-free graphs, shedding light on the chromatic properties of graph structures devoid of triangles. Nešetřil and Ossona de Mendez (2009) offer a comprehensive survey on colorings and homomorphisms in minor closed classes, enriching the understanding of combinatorics. Krivelevich and Sudakov (2006) delve into pseudo-random graphs, contributing to the exploration of their properties and applications in combinatorics. Brandt, De Marco, and Rautenbach (2011) investigate fractional chromatic numbers and local constraints, advancing discussions on graph theory and constraints.”

The subsequent studies explore diverse facets, including the fractional chromatic number of Mycielski graphs (Fan & Liu, 2013), random graphs (Montellano-Ballesteros & Rodríguez-Velázquez, 2016), Cartesian products of graphs (Czap & Havet, 2015), and upper bounds on the chromatic number of graphs (Škrekovski & Vizing, 2007).

“Notable contributions include the examination of the fractional chromatic number of line graphs resulting from Mycielski's construction (Zhu, Xu, & Ma, 2017) and the exploration of power graphs of cycle graphs (Erwin & Zhu, 2014).

Practical insights are offered by Pemmaraju and Skiena (2003) in "Computational Discrete Mathematics," while foundational knowledge is provided by Bondy and Murty's "Graph Theory" (2008). The literature collectively enriches our understanding of graph coloring properties, spanning theoretical foundations to practical applications.

2. PROPOSED DEFINITIONS

Definition 1: Fractional Colorings

We now generalize the idea of a proper coloring to that of a fractional coloring (or a set coloring), which allows us to define a graph’s fractional chromatic number, denoted $\chi_F(G)$, which can assume non-integer values.

Given a graph, integers $0 < b \leq a$, and a set of a colors, a proper a/b −coloring is a function that assigns to each vertex a set of b distinct colors, in such a way that adjacent vertices are assigned disjoint sets. Thus, a proper n-coloring is equivalent to a proper $n/1$ −coloring. The definition of a fractional coloring can also be formalized by using graph homomorphisms.

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To this end, we define another family of graphs, the *Kneser graphs*. For each ordered pair of positive integers (a, b) with $a \geq b$, we define a graph $K_{a:b}$ As the vertex set of $K_{a:b}$, we take the set of all b-element subsets of the set $\{1, \ldots, a\}$.

Two such subsets are adjacent in $K_{a:b}$ if and only if they are disjoint. Note that $K_{a:b}$ is an empty graph (i.e., its edge set is empty) unless $a \geq 2b$. Just as a proper n-coloring of a graph G can be seen as a graph homomorphism from G to the graph K_n, so a proper a/b-coloring of G can be seen as a graph homomorphism from G to $K_{a:b}$.

Definition 2: Fractional Chromatic Number:

The fractional chromatic number of a graph, $\chi_F(G)$, is the infimum of all rational numbers a/b such that there exists a proper a/b-coloring of G. From this definition, it is not immediately clear that $\chi_F(G)$ must be a rational number for an arbitrary graph. In order to show that it is, we will use a different definition of fractional coloring, but first, we establish some bounds for $\chi_F(G)$ based on our current definition.

We can get an upper bound on the fractional chromatic number using the chromatic number. If we have a proper n-coloring of G, we can obtain a proper coloring for any positive integer b by replacing each individual color with b different colors. Thus, we have $\chi_F(G) \leq \chi(G)$, or in terms of homomorphism's, we can simply note the existence of a homomorphism from K_n to$K_{nb:b}$ (namely, map i to the set of $j \equiv i \ (mod\ n)$).

To obtain one lower bound on the fractional chromatic number, we note that a graph containing an n-clique has a fractional coloring with b colors on each vertex only if we have at least $n \cdot b$ colors to choose from; inother words, $\omega(G) \leq \chi_F(G)$. Just as with proper colorings, we can obtain another lower bound from the independence number.

Since each color in a fractional coloring is assigned to an independent set of vertices (the fractional color class), we have $|V(G)| \cdot b \leq \alpha(G) \cdot a$, or $\frac{|V(G)| \cdot}{\alpha(G)} \leq \chi_F(G)$.

Another inequality, which will come in handy later, regards fractional colorings of subgraphs. A graph H is said to be a subgraph of G if $V(H) \subseteq V(G)$and $E(H) \subseteq E(G)$. Notice that if H is a subgraph of G, then any proper a/b-coloring of G, restricted to $V(H)$, is a proper a/b −coloring of H. This tells us that$\chi_F(H) \leq \chi_F(G)$."

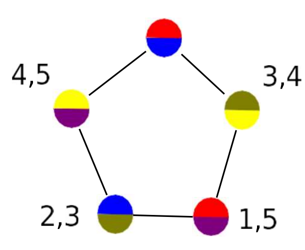

Figure 1: The graph C_5 with a proper 5/2-coloring

Definition 3: Fractional Colorings In Terms Of Independent Sets

Now we introduce an alternative definition of fractional colorings, one expressed in terms of independent sets of vertices. This definition is somewhat more general than the previous one, and we will see how fractional colorings, understood as homomorphisms, can be consistently reinterpreted in terms of independent sets. This new characterization of fractional colorings will lead us to some methods of computing a graph's fractional chromatic number.

"Let $I(G)$ denote the set of all independent sets of vertices in $V(G)$, and for $u \in V(G)$, we let $I(G,u)$ denote the set of all independent sets of vertices containing u. In this context, a **fractional coloring** is a mapping $f : I(G) \to [0,1]$ with the property that, for every vertex u,$\sum_{J\in I(G,u)} f(J) \geq 1$.The sum of the function values over all independent sets are called the weight of the fractional coloring.

Definition 4: Fractional Chromatic Number

The fractional chromatic number of G is the infimum of the weights of all possible fractional colorings of G.

It may not be immediately clear that this definition has anything to do with our previous definition, in terms of graph homomorphisms. To see the connection, consider first a graph G with a proper coloring. Each color class is an independent set belonging to $I(G)$. We define $f : I(G) \to [0,1]$ mapping each color class to 1 and every other independent set to 0. Since each vertex falls in one color class, we obtain

$$\sum_{J\in I(G,u)} f(J) \geq 1$$

for each vertex u. The weight of this fractional coloring is simply the number of colors.

Next, suppose we have a graph G with a proper a/b coloring as defined above, with a b-element set of colors associated with each vertex. Again, each color determines a color class, which is an independent set. If we define a function that sends each color class $\frac{1}{b}$ to and every other independent set to 0, then again, we have for each vertex u, $\sum_{J\in I(G,u)} f(J) \geq 1$. , so we have a fractional coloring by our new definition, with weight a/b. Finally, let us consider translating from the new definition to the old one. Suppose we have a graph G and a function f mapping from $I(G)$ to $[0,1] \cap Q$. (We will see below why we are justified in restricting our attention to rational valued functions.) Since the graph G is finite, the set $I(G)$ is finite, and the image of the function f is a finite set of rational numbers. This set of numbers has a lowest common denominator, b. Now suppose we have an independent set I which is sent to the number m/b. Thus, we can choose m different colors, and let the set I be the color class for each of them. Proceeding in this manner, we will assign at least b different colors to each vertex, because of our condition that for all u, $\sum_{J\in I(G,u)} f(J) \geq 1$. If some vertices are assigned more than b colors, we can ignore all but b of them, and we have a fractional coloring according to our old definition - if the weight of f is a/b, and we do not ignore any colors completely, then we will have obtained a proper a/b coloring. If some colors are ignored, then we actually have a proper d/b fractional coloring, for some $d < a$.

Definition 5: Fractional Chromatic Number as a Linear Program

The usefulness of this new definition of fractional coloring and fractional chromatic number in terms of independent sets is that it leads us to a method of calculation using the tools of linear programming. To this end, we will construct a matrix representation of a fractional coloring.

For a graph G, define a matrix $A(G)$, with columns indexed by $V(G)$ and rows indexed by $I(G)$. Each row is essentially the characteristic function of the corresponding independent set, with entries equal to 1 on columns corresponding to vertices in the independent set, and 0 otherwise.

Now let f be a fractional coloring of G and let $y(G)$ be the vector indexed by $I(G)$. with entries given by f. With this notation, and letting 1 denote the all 1's vector, the inequality $y(G)^T A(G) \geq 1^T$ expresses the condition that

$$\sum_{J \in I(G,u)} f(G) \geq 1$$

for all $u \in V(G)$.

In this algebraic representation of a fractional coloring, the determination of fractional chromatic number becomes a linear programming problem. The entries of the vector $y(G)$ are a set of variables, one for each independent set in $V(G)$, and our task is to minimize the sum of the variables (the weight of the fractional coloring), subject to the set of constraints that each entry in the vector $y(G)^T A(G)$ be greater than 1, and that each variable be in the interval [0, 1]."

This amounts to minimizing a linear function within a convex polyhedral region in n-dimensional number of linear inequalities, where $n = |I(G)|$. This minimum must occur at a vertex of the region. Since each hyper plane forming a face of the region is determined by a linear equation with integer coefficients, then each vertex has rational coordinates, so our optimal fractional coloring will indeed take on rational values, as promised.

" The regular, integer chromatic number, can be calculated with the same linear program by restricting the values in the vector $y(G)$ to 0 and 1. This is equivalent to covering the vertex set by independent sets that may only have weights of 1 or 0.

Although polynomial time algorithms exist for calculating optimal solutions to linear programs, this is not the case for integer programs or 0-1 programs. In fact, many such problems have been shown to be NP-hard. In this respect, fractional chromatic numbers are easier to calculate than integer chromatic numbers.

Definition 6: Fractional Cliques

The linear program that calculates a graph's fractional chromatic number is the *dual* of another linear program, in which we attempt to maximize the sum of elements in a vector $x(G)$, subject to the constraint $A(G)x(G) \leq 1$. We can pose this maximization problem as follows: we want to define a function $h : V(G) \rightarrow [0, 1]$, with the condition that, for each independent set in I(G), the sum of function values on the vertices in that set is no greater than 1. Such a function is called a fractional clique, the dual concept of a fractional coloring. As with fractional colorings, we define the weight of a fractional clique to be the sum of its values over its domain. The supremum of weights of fractional cliques defined for a graph is a parameter, $\omega_F(G)$, the fractional clique number.

Just as we saw a fractional coloring as a relaxation of the idea of an integer coloring, we would like to understand a fractional clique as a relaxation of the concept of a integer clique to the rationals (or reals). It is fairly straightforward to understand an ordinary clique as a fractional clique: we begin by considering a graph G, and a clique, $C \subseteq V(G)$. We can define a function $h : V(G) \rightarrow [0, 1]$ that takes on the value 1 for each vertex in C and 0 elsewhere. This function satisfies the condition that its values sum to no more than 1 over each independent set, for no independent set may intersect the clique C in more than one vertex. Thus the function is a fractional clique, whose weight is the number of vertices in the clique.

Since an ordinary n-clique can be interpreted as a fractional clique of weight n, we can say that for any graph $G, \omega(G) \leq \omega_F(G)$.

3. PROPOSED THEOREMS

Definition 3.1: Mycielski's Graphs

We have noted that the fractional clique number of a graph G is bounded from below by the integer clique number, and that it is equal to the fractional chromatic number, which is bounded from above by the integer chromatic number. In other words,

$$\omega(G) \leq \omega_F(G) = \chi_F(G) \leq \chi(G).$$

Given these relations, one natural question to ask is whether the differences $\omega_F(G) - \omega(G)$ and $\chi(G) - \chi_F(G)$ can be made arbitrarily large. We shall answer this question in the affirmative, by displaying a sequence of graphs for which both differences increase without bound.

THEOREM 3.2 The Mycielski Transformation

The sequence of graphs we will consider is obtained by starting with a single edge K_2, and repeatedly applying a graph transformation, which we now define. Sup- pose we have a graph G, with $V(G) = \{v1, v2, \ldots, vn\}$. The **Mycielski transformation** of G, denoted $\mu(G)$, has for its vertex set the set $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency,

we put

$x_i \sim x_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,

$x_i \sim y_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,

and $y_i \sim z$ in $\mu(G)$ for all $i \in \{1, 2, \ldots, n\}$.

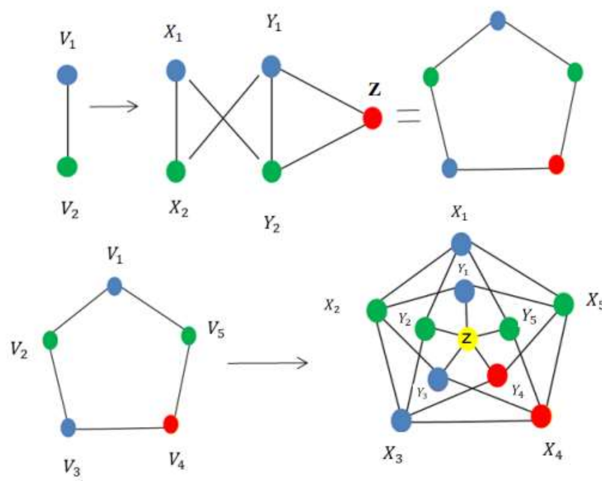

Figure 2: The Mycielski Transformation

The theorem that we shall prove states that this transformation, applied to a graph G with at least one edge, results in a graph $\mu(G)$ with

(a) $\omega(\mu(G)) = \omega(G)$,

(b) $\chi(\mu(G)) = \chi(G) + 1$, and

(c) $\chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}$

Finding solution for the theorem:

We prove each of the three statements above in order:

I. Proof of $\omega(\mu(G)) = \omega(G)$,

First we note that the vertices $x_1, x_2, \ldots, x_n$ form a subgraph of $\mu(G)$ which is isomorphic to G. Thus, any clique in G also appears as a clique in $\mu(G)$, so we have that $\omega(\mu(G)) \geq \omega(G)$. To obtain the opposite inequality, consider cliques in $\mu(G)$. First, any clique containing the vertex z can contain only one other vertex, since z is only adjacent to the y vertices, none of which are adjacent to each other.

Now consider a clique $\{x_{i(1)}, \ldots, x_{i(r)}, y_{j(1)}, \ldots, y_{j(s)}\}$. From the definition of the Mycielski transformation, we can see that the sets $\{i(1), \ldots, i(r)\}$ and $\{j(1), \ldots, j(s)\}$ are disjoint, and that the set $\{v_{i(1)}, \ldots, v_{i(r)}, v_{j(1)}, \ldots, v_{j(s)}\}$ is a clique in G. Thus, having considered cliques with and without vertex z, we see that for every clique in $\mu(G)$, there is a clique of equal size in G, or in other words, $\omega(\mu(G)) \leq \omega(G)$.Combining these inequalities, we have $\omega(\mu(G)) = \omega(G)$, as desired.

II. Proof of $\chi(\mu(G)) = \chi(G) + 1$

Suppose we have that $\chi(G) = k$. We must show that $\chi(\mu(G)) = k + 1$. First, we shall construct a proper k+1-coloring of $\mu(G)$, which will show that $\chi(\mu(G)) \leq k + 1$. Suppose that f is a proper k-coloring of G, understood as a mapping $f : V(G) \to \{1, \ldots, k\}$. We define a proper $k + 1$-coloring, $h : V(\mu(G)) \to \{1, \ldots, k + 1\}$ as follows. We set $h(x_i) = h(y_i) = f(v_i)$, for each $i \in \{1, \ldots, n\}$. Now set $h(z) = k + 1$. From the way $\mu(G)$ was constructed, we can see that this is a proper coloring, so we have $\chi(\mu(G)) \leq k + 1$.

For the inequality in the opposite direction, we will show that, given any coloring of $\mu(G)$, we can obtain a coloring of G with one color fewer. Since the chromatic number of G is k, this will show that $\chi(\mu(G)) - 1 \geq k$, or equivalently, $\chi(\mu(G)) \geq k + 1$. So, consider a proper coloring h of $\mu(G)$. We define a function f on the vertices of G as follows: $f(v_i) = h(x_i)$ if $h(x_i) \neq h(z)$, and $f(v_i) = h(y_i)$ if $h(x_i) = h(z)$. From the construction of $\mu(G)$, it should be clear that this is a proper coloring. It does not use the color $h(z)$, so it uses one color fewer than h uses, and thus we have that $\chi(\mu(G)) \geq k + 1$. Again, combining our two inequalities, we obtain $\chi(\mu(G)) = \chi(G) + 1$, as desired.

III. Proof of $\boldsymbol{\chi_F}(\mu(G)) = \boldsymbol{\chi_F(G)} + \frac{\mathbf{1}}{\boldsymbol{\chi_F(G)}}$

Now we will show that $\chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}$ or in other words, we want to show that if $\chi_F(G) + \frac{a}{b}$,then $\chi_F(\mu(G)) = \chi_F(G) + \frac{a^2+b^2}{ab}$ Our strategy will be to construct a fractional coloring and a fractional clique on $\mu(G)$, each with the appropriate weight, and then invoke our strong duality result. Suppose we have a proper a/b-coloring of G, understood in terms of sets of colors assigned to each vertex. Thus, each vertex in G is assigned some subset of b colors out of a set of size a.

We suppose, somewhat whimsically, that each of the a colors has a "offspring", b of them "male" and $a - b$ of them "female". Taking all of these offspring as distinct, we have obtained a^2 offspring colors. To these, add a set of b^2 new colors, and we have a total of $a^2 + b^2$ colors with which to color the vertices of $\mu(G)$.

We assign them as follows. To each of vertex x_i, we assign all the offspring of all the colors associated with vertex v_i, a offspring of each of b colors for a total of ab colors. To each vertex y_i, we assign all the female offspring of all the colors associated with v_i (there are $b(a - b)$), and all of the new colors (there are b^2). To vertex z, we assign all of the male offspring of all the original colors, which is ab distinct colors. We see that we have assigned ab colors to each vertex, and it is easy to check that the coloring is a proper $(a^2 + b^2)/ab$-coloring of $\mu(G)$.The existence of this coloring proves that $\chi_F(\mu(G)) \leq \frac{a^2+b^2}{ab}$.

Our final, and most complicated, step is to construct a fractional clique of weight $\omega_F(G) + \frac{1}{\omega_F(G)}$ on μ(G). Suppose we have a fractional clique on G that achieves the optimal weight $\omega_F(G)$. Recall that this fractional clique is understood as a mapping $f : V(G) \to [0, 1]$ which sums to at most 1 on each independent set, and whose values all together sum to $\omega_F(G)$. Now we define a mapping $g : V(\mu(G)) \to [0, 1]$ as follows:"

$$\mathrm{g}(x_i) = \left(1 - \frac{1}{\omega_F(G)}\right) f(v_i)$$

$$\mathrm{g}(y_i) = \left(\frac{1}{\omega_F(G)}\right) f(v_i)$$

$$\mathrm{g}(z) = \left(\frac{1}{\omega_F(G)}\right)$$

We must show that this is a fractional clique. In other words, we must establish that it maps its domain into [0, 1], and that its values sum to at most 1 on each independent set in $\mu(G)$. The codomain is easy to establish the range of f lies between 0 and 1, and since $\omega_F(G) \geq \omega(G) \geq 2 > 1$, then $0 < \frac{1}{\omega_F(G)} < 1$. $\omega F(G) < 1$. Thus each expression in the definition of g yields a number between 0 and 1. It remains to show that the values of g are sufficiently bounded on independent sets.

We introduce a notation: for $M \subseteq V(G)$, we let $x(M) = \{x_i|v_i \in M\}$ and$y(M) = \{y_i|v_i \in M\}$. Now we will consider two types of independent sets in $\mu(G)$: those containing z and those not containing z.

“Any independent set $S \subseteq V(\mu(G))$ that contains z cannot contain any of the y_i vertices, so it must be of the form $S = \{z\} \cup x(M)$ for some independent set M in $V(G)$. Summing the values of g over all vertices in the independent set, we obtain:

$$\sum_{v\epsilon S} g(v) = g(z) + \sum_{v\epsilon x(M)} g(v)$$

$$= \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right)\sum_{v\epsilon M} f(v) \leq \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) = 1$$

Considering an independent set $S \subseteq V(\mu(G))$ with $z \notin S$. we can express $S = x(M) \cup y(N)$ for some subsets of $V(G)$, M and N. It is known that M is an independent set. Since S is independent, no vertex in $y(N)$ is adjacent to any vertex in $x(M)$. Therefore, we can represent N as the union of two sets A and B, where $A \subseteq M$ and none of the vertices in B are adjacent to any vertex in M. Now, the values of g over the vertices in $S=x(M) \cup y(N)=x(M) \cup y(A) \cup y(B)$ can be summed:

$$\sum_{v\epsilon S} g(v) = \left(1 - \frac{1}{\omega_F(G)}\right)\sum_{v\epsilon M} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon N} f(v)$$

$$= \left(1 - \frac{1}{\omega_F(G)}\right)\sum_{v\epsilon M} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon A} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon B} f(v)$$

$$\leq \left(1 - \frac{1}{\omega_F(G)}\right)\sum_{v\epsilon M} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon M} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon B} f(v) = \sum_{v\epsilon M} f(v) + \left(\frac{1}{\omega_F(G)}\right)\sum_{v\epsilon B} f(v)$$

The initial two equalities mentioned above involve breaking down the sum into sub-sums that correspond to specific subsets. The inequality is justified by the inclusion of A within M, and the last equality results from simplification. At this point, demonstrating that the final expression is less than or equal to 1 becomes sufficient.

In examining the subgraph H induced by B in G, with a fractional chromatic number of r/s, let's assume a proper r/s-coloring of H. The color classes of a fractional coloring represent independent sets, resulting in r independent sets denoted as $C_1, \ldots, C_r$, in $V(H) = B$. Then, each set C_i is not only independent in H but also independent in G. Furthermore, the union of C_i and M remains independent in G, as C_i is a subset of B. This interplay emphasizes the preservation of independence properties through the subgraph H and its relationship with G.

For each i, we note that f is a fractional clique on G, and sum over the independent set $C_i \cup M$ to obtain:

$$\sum_{v\epsilon M} f(v) + \sum_{v\epsilon\ \ i} f(v) \leq 1$$

Summing these inequalities over each C_i, we get:

$$r\sum_{v\epsilon M} f(v) + s\sum_{v\epsilon B} f(v) \le r$$

The second term on the left side of the inequality results because each vertex in B belongs to s different color classes in our proper r/s-coloring. Now we divide by r to obtain:

$$\sum_{v\epsilon M} f(v) + \frac{r}{s}\sum_{v\epsilon B} f(v) \le 1$$

Since r/s is the fractional chromatic number of H, and H is a subgraph of G, we can say $\frac{r}{s} \le \ \chi_F(G) = \omega_F(G)$, or equivalently, $\frac{1}{\omega_F(G)} \le \frac{r}{s}$. Thus

$$\sum_{v\epsilon M} f(v) + \frac{1}{\omega_F(G)}\sum_{v\epsilon B} f(v) \le \sum_{v\epsilon M} f(v) + \frac{r}{s}\sum_{v\epsilon B} f(v) \le 1$$

as required. We have shown that the mapping g that we defined is indeed a fractional clique on μ(G). We now check its weight

$$\sum_{v\epsilon V(\mu(G))} g(v) = \sum_{i=1}^{n} h(x_i) + \sum_{i=1}^{n} h(y_i) + h(z)$$

$$= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v\epsilon V(G)} f(v) + \left(\frac{1}{\omega_F(G)}\right) \sum_{v\epsilon V(G)} f(v) + \left(\frac{1}{\omega_F(G)}\right)$$

$$= \sum_{v\epsilon V(G)} f(v) + \left(\frac{1}{\omega_F(G)}\right)$$

$$\omega_F(G) + \left(\frac{1}{\omega_F(G)}\right) = \ \chi_F(G) + \frac{1}{\chi_F(G)}$$

This is the required weight, so we have constructed a fractional coloring and a fractional clique on μ(G), both with weigh $\chi_F(G) + \frac{1}{\chi_F(G)}$. We can now write the inequality

$$\chi_F(\mu(G)) \le \ \chi_F(G) + \frac{1}{\chi_F(G)} \le \ \omega_F(G)$$

and invoke strong duality to declare the terms at either end equal to each other, and thus to the middle term.

Analysis of the result:

Applying our theorem, first to clique numbers, we see that $\omega(G_n) = 2$ for all n. Considering chromatic numbers, we have $\chi(G_2) = 2$ and $\chi(G_{n+1}) = \chi(G_n)+1$; thus $\chi(G_n) = n$ for all n.

Finally, the now that we have established a theorem detailing the impact of the Mycielski transformation on the clique number (ω), chromatic number (χ), and fractional chromatic number (χ_F), let us delve into a practical application by iteratively applying this transformation to generate a sequence of graphs $\{G_n\}$, where $G_{n+1} = \mu(G_n)$ for $n > 2$. To initiate this sequence, we begin with G_2, which consists of a single edge, K_2, leading to ω(G) = χ_FG) = χ(G) = 2.

Applying our theorem sequentially, starting with clique numbers, we observe that $\omega(G_n)$ remains constant at

2 for all n. Shifting our focus to chromatic numbers, we find that $\chi(G_2) = 2$, and subsequently, $\chi(G_{n+1}) = \chi(G_n)+1$. Consequently, the chromatic number of G_n is precisely n for all n.

Finally, the fractional chromatic number of G_n follows a recurring sequence $\{a_n\}$ for n ∈ {2, 3, ...}, determined by the recurrence relation: $a_2 = 2$ and $a_{n+1} = a_n + \frac{1}{a_n}$. This recurrence governs the evolution of the fractional chromatic number across the graph sequence.

This sequence has been studied, and it is known that for all n:

$$\sqrt{2n} \leq a_n \leq \sqrt{2n + \frac{1}{2}\ln \ n}$$

The sequence a_n, determined by the Mycielski transformation on graphs, grows indefinitely, yet at a slower rate than any sequence of the form n^r for $r > \frac{1}{2}$. Consequently, the disparities between the fractional clique number and the clique number, as well as the chromatic number and the fractional chromatic number, exhibit unbounded growth. This highlights the dynamic and unbounded nature of these fundamental graph parameters through successive Mycielski transformations.

IV. APPLICATION OF THE FRACTIONAL CHROMATIC NUMBER AND MYCIELSKI GRAPHS

"The fractional chromatic number and Mycielski graphs, while primarily studied for their theoretical properties in graph theory, may not have direct and obvious real-life applications. However, the concepts and methods associated with these mathematical constructs can find indirect applications in various areas. Here are some potential scenarios where fractional chromatic numbers and Mycielski graphs could be relevant:"

1. Communication Networks:

Fractional chromatic numbers can be related to channel assignment problems in wireless communication networks. Efficiently assigning frequencies to channels while minimizing interference is crucial. Mycielski graphs, through their connection to the fractional chromatic number, might offer insights into optimizing channel assignments for better communication.The optimization problem can be formulated with the goal of minimizing interference in wireless communication networks. Let's define the following terms:

- G: The original communication network graph.
- n: The number of vertices in G.
- E: The set of edges in G.
- $\chi_f(G)$: The fractional chromatic number of G.

The objective is to find an optimal channel assignment that minimizes interference, and this can be expressed as:

Minimize $\chi_f(G)$

Subject to the constraint that for each edge $e = (u, v) \in E$, the colors assigned to u and v must be different, ensuring no interference between adjacent vertices:

Subject to for Subject to $c(u) \neq c(v)$ for $e = (u, v) \in E$

Here, $c(u)$ represents the color assigned to vertex u.

Mycielski's construction involves creating a new graph $\mu(G)$ from G, and the fractional chromatic number of the Mycielski graph is denoted as $\chi_f(\mu(G))$. The transformation preserves certain properties of the original graph while influencing its chromatic characteristics.The equation representing the optimization problem using Mycielski graphs and fractional chromatic numbers becomes:

Minimize $\chi_f(\mu(G))$

Subject to the constraint that for each edge ∈ $e = (u, v) \in E$, the colors assigned to u and v must be different:

Subject to for Subject to $c(u) \neq c(v)$ for $e = (u, v) \in E$

By solving this optimization problem, one can gain insights into an efficient channel assignment strategy that minimizes interference and enhances communication network performance.

2. Resource Allocation in Networks:

In network design, determining the fractional chromatic number could be related to optimizing resource allocation, where nodes represent resources and edges denote conflicts or dependencies. Mycielski graphs might aid in simplifying the analysis of resource allocation problems with complex connectivity requirements.

3. Scheduling and Timetabling:

Fractional chromatic numbers can be associated with scheduling problems, where tasks or events with conflicting resource requirements need to be assigned time slots. Mycielski graphs could potentially assist in modeling and optimizing scheduling scenarios with intricate dependencies.

4. Circuit Design:

In electronic circuit design, fractional chromatic numbers might be linked to the allocation of limited resources such as power, area, or signal integrity. Mycielski graphs could offer insights into optimizing circuit layouts while considering conflicting constraints.

5. Graph Coloring in Map Labeling:

Fractional chromatic numbers have applications in map labeling problems, where regions on a map need to be colored with minimal conflict. Mycielski graphs may aid in understanding the inherent complexities of map labeling and finding efficient labeling strategies.

6. Biological Networks:

The fractional chromatic number and Mycielski graphs could find applications in the study of biological networks, such as gene regulatory networks or protein interaction networks. Understanding the dependencies and conflicts within these networks might benefit from graph coloring concepts.

7. Fault-Tolerant Systems:

Mycielski graphs and related constructions can be applied to fault-tolerant system design. Understanding how certain properties are preserved or modified under Mycielski transformations can help in designing systems resilient to faults or failures.

8. Communication Protocols:

In the design of communication protocols, particularly in distributed systems, Mycielski graphs can be used to model and analyze the communication patterns. Theoretical insights gained from Mycielski transformations may guide the development of more efficient and reliable communication schemes.

9. Traffic Flow Modeling:

Mycielski graphs might find applications in traffic flow modeling where understanding the interactions between different components of a transportation network is crucial.Theoretical results about Mycielski transformations can inform the design of efficient traffic management systems.

10. Quantum Computing:

In the emerging field of quantum computing, where graph-based algorithms play a significant role, Mycielski graphs can be explored for their quantum counterparts and their impact on quantum algorithm design.

11. Epidemiology and Disease Spread:

Mycielski graphs can be utilized to model the spread of diseases in populations. Theoretical insights gained from graph transformations may contribute to understanding how interventions or preventive measures impact the spread of diseases.

12. Data Compression and Representation:

Mycielski graphs have been used in data compression schemes and information representation. The structure-preserving nature of Mycielski transformations can be exploited to represent complex systems more compactly.

13. Cryptography:

In certain cryptographic protocols, the structure-preserving nature of Mycielski transformations can be utilized for secure communication or key exchange.

14. Robotics and Sensor Networks:

Mycielski graphs can be employed in modeling communication and interaction patterns in robotics and sensor networks. This can be useful in optimizing the deployment and coordination of robotic systems or sensors.

15. Supply Chain Optimization:

Mycielski graphs may be applicable in modeling and optimizing supply chain networks. Theoretical insights from graph transformations can guide the design of efficient and resilient supply chain structures.

16. Internet of Things (IoT):

In IoT systems, where various devices communicate and interact, Mycielski graphs could be used to model the connectivity and relationships between IoT nodes. This can contribute to the design of robust and efficient IoT architectures.

V. CONCLUSION

In conclusion, the study delves into the intricate relationships between the clique number (ω), chromatic number (χ), and fractional chromatic number (χ_F) of graphs under the transformative Mycielski operation. The main result provides a comprehensive understanding of how these graph parameters evolve through iterative applications of the Mycielski transformation, yielding a sequence of graphs with distinct properties.

The transformation showcases its impact by maintaining the clique number constant, steadily increasing the chromatic number, and expressing a recurring fractional chromatic number pattern. This finding unveils the dynamic nature of graph structures as they undergo the Mycielski process, shedding light on the interplay between fundamental graph properties."

Furthermore, the application of the main result in generating a sequence of graphs exemplifies the versatility and significance of the Mycielski transformation in graph theory. The derived sequence, characterized by consistent clique numbers and escalating chromatic numbers, offers a valuable framework for exploring various graph-related phenomena.

Overall, the study contributes not only to the theoretical understanding of graph transformations but also paves the way for potential applications and extensions in graph theory and related fields. The intricate balance and interdependence revealed between ω, χ, and χ_F provide a foundation for future investigations into graph evolution and dynamic structural changes.

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