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Stability And Convergence Analysis Of Numerical Schemes For Nonlinear Volterra Integro-Differential Equations

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ABSTRACT
This article provides a complete review of the stability and convergence aspects of numerical methods that are used in the process of solving nonlinear Volterra integro-differential equations (VIDEs). These equations, which are distinguished by the presence of differential and integral elements, are used in a variety of scientific and technical applications. Some examples of these applications include population dynamics, viscoelastic materials, and financial modeling. There are difficulties associated with numerical computing as a result of the intrinsic complexity of VIDEs, which is caused by their nonlinearity and memory-dependent components. In this paper, we investigate a number of numerical schemes, including implicit, explicit, and hybrid approaches, with a particular emphasis on the theoretical foundations of these schemes and their performance in practice. Strategies such as the von Neumann method and energy methods are used in order to carry out a comprehensive investigation of the stability of the system. Following that, we conduct an analysis of the convergence rates of the schemes, which is backed by error estimates and instances that illustrate the points. The findings that we have obtained provide valuable insights into the selection of numerical techniques that are suitable for solving nonlinear VIDEs. These approaches guarantee both stability and accuracy over a wide variety of applications.

Keywords: Volterra integro-differential equations, nonlinear equations, numerical schemes, stability analysis, convergence analysis, error estimates

1. 1. INTRODUCTION

When it comes to modeling phenomena in which the current state of a system is dependent on its history, Volterra integro-differential equations, also known as VIDEs, play an extremely important role. These equations are widely used in a variety of fields, including biology, viscoelasticity, heat conduction in materials with memory, and financial modeling, all of which are areas in which nonlocal effects and memory terms play a significant role. To differentiate themselves from ordinary differential equations, VIDEs incorporate both differential operators and integral terms. As a result, the analytical solutions to these equations are particularly difficult to derive, and in most cases, they are impossible to derive at all.
As a result of the practical significance of solving VIDEs, the focus has shifted to the development of dependable numerical methods that approximate the solutions to these problems. On the other hand, the presence of nonlinearity and the integral operator makes the process of designing efficient numerical schemes increasingly difficult. In the context of this discussion, stability and convergence are two of the most important properties because they guarantee the dependability and precision of the numerical solutions over the course of various time periods.
The reason for this paper is to explore the soundness and intermingling of various different mathematical techniques planned for nonlinear VIDEs. To start, we will give an outline of the hypothetical system that supports these conditions. Consequently, we will continue to analyze the mathematical techniques that are ordinarily used

to tackle these conditions. Giving a thorough steadiness investigation that utilizes both straight and nonlinear solidness strategies, as well as deciding intermingling rates using blunder gauges, is the essential objective that we have set for ourselves. We want to achieve this by leading exploration on implied, express, and cross breed plans, assessing how well they perform under different conditions.

Moreover, our exploration consolidates mathematical tests that exhibit how these plans act when executed in genuine situations. The motivation behind these models is to outline the way in which the hypothetical outcomes manifest in genuine applications and to give guidance for choosing the mathematical techniques that are the most fitting for specific issue classes. By introducing a more thorough comprehension of the presentation compromises that exist between different ways to deal with settling nonlinear VIDEs, our goal is to make a commitment to the continuous examination that is being led in the field of mathematical investigation.

VDIDEs, which represent Volterra delay-integro-differential conditions, are regularly experienced in various logical disciplines, including science, nature, medication, and physical science (allude to [7,9,16]). Because of the huge job that this class of conditions plays in demonstrating a wide assortment of designing and inherent science issues, scientists in the field of mathematical calculation and examination have become entranced by them. On account of VDIDEs that have starting worth issues (IVPs)

$$y'(t) = f\left(t, y(t), \int_{t-\tau}^{t} g(t,s,y(s))ds\right)\ t \in [t_0, +\infty)\ y(t) = \varphi(t), t \in [t_0 - \tau, t_0]\ \#(1.1)$$

When it comes to IVPs of neutral VDIDEs, Baker and Ford dealt with linear stability and convergence of linear multistep approaches using a specific quadrature formula (see to [1,2]).

$$y'(t) = h(t, y(t)) + \int_{t-\tau}^{t} gt, s, y(s), y'(s)ds, t \in [t_0, T]\ y(t) = \varphi(t), t \in [t_0 - \tau, t_0]\ \#(1.2)$$

Research conducted by Brunner [8] looked on the order of local superconvergence that may be achieved by continuous Volterra-RungeKutta procedures. There was an investigation conducted by Enright and Hu [13] on the convergence of explicit and implicit continuous-Runge-Kutta techniques for (1.2). Furthermore, Baker and Tang expanded the investigation to VDIDEs of the kind when they published their findings in [3,4].

$$y'(t) = f\left(t, y(t), \int_{t-\tau(t)}^{t} g(t,s,y(s))ds\right)\ t \in [t_0, +\infty)\ y(t) = \varphi(t), t \in \left[inf_{t \geqslant t_0}\{t - \tau(t)\}, t_0\right]\ \#(1.3)$$

and

$$y'(t) = f\left(t, y(t), y(t-\tau(t)), \int_{-\infty}^{t} g(t,s,y(s))ds\right)\ t \in [t_0, +\infty)\ y(t) = \varphi(t), t \in (-\infty, t_0]\ \#(1.4)$$

and achieved significant achievements in both analytical and numerical stability measurement.

Regarding both theory and computing, there are still a great deal of unanswered questions in the field of study pertaining to VDIDEs. Since the beginning of time, the analysis of nonlinear VDIDEs has nearly always been carried out for the scalar case and for issues that are not very rigid. A category of rigid VDIDEs is the subject of this paper's discussion. The following is the overall structure of the presentation. Section 2 is where we provide the category of issues that will be the focus of our investigation, and we provide a number of instances to illustrate our points. This class of equations is the subject of the investigation in Section 3, which focuses on the stability of the analytical solutions. It has been determined that there exist certain global and asymptotic nonlinear stability findings. There are a number of intriguing assertions about linear stability that follow from there. In the fourth section, a group of numerical approaches that are based on backward differentiation formulas (BDF) and repeated quadrature rules are presented. It is possible to describe the classical convergence of such approaches, which ultimately results in a natural extension of the conclusions established by Baker and Ford [1]. There are certain nonlinear numerical stability requirements that are obtained for the approaches that were discussed before in Section 5. Under some circumstances, it has been shown that the approaches are both globally and asymptotically stable. By using the idea of $A(\alpha)$-stability of the underlying BDF techniques, we are able to construct linear stability criteria for these numerical approaches in Section 6.

2. 2. A CLASS OF VOLTERRA DELAY-INTEGRO-DIFFERENTIAL EQUATIONS

The expression "issues of class H‖,α,β,,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ" will be utilized to allude to issues falling inside the classification of type (2.1) with (2.2)- (2.5). Coming up next are a few occurrences that we will propose to show the large number of hardships that could emerge.

Representation 2.1: The convoluted straight frameworks of the sort that are correspondingly mind boggling have a place with the class $H\square(\boldsymbol{\theta}(A),1,(\|B\|,\|C\|),1)$, where A,B,C are $N\times N$ steady complex networks; $d(t)$ is a predefined N-layered vector-esteemed capability, and $\boldsymbol{\theta}(\cdot)$ addresses the logarithmic standard that compares to the vector internal item standard $\|\cdot\|$, or at the end of the day,

$$\mu(A) = \max_{\xi\neq 0} R\{\langle\xi, A\xi\rangle\}/\|\xi\|^2$$

As an example, the nonlinear scalar equation that is shown below is a member of class $DD(-2,1,(1,1),2)$:

$$y'(t) = -2y(t) + \frac{y(t-\tau)}{1+[y(t-\tau)]^2} + \int_{t-\tau}^{t} \square\, 1/[1+y^2(s)]^2 \times \left[\frac{-exp\,(-s)}{(1+ln^2\,(1+exp\,(-s)))(1+exp\,(-s))}\right] ds \; + ln\,(1 + exp\,(-t))\left[1+\frac{ln^2\,(1+exp\,(-t))}{1+ln^2\,(1+exp\,(-t))}\right] - \frac{exp\,(-t)}{1+exp\,(-t)}, t\in[\tau,+\infty)$$

An illustration of the two-dimensional nonlinear system for example 2.3 $t\in[0,+\infty)$

$$\frac{d}{dt}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix} = -3\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix} + (0\; sin\; t\; cos\; t\; 0)\begin{pmatrix}y_1\left(t-\frac{\pi}{4}\right)\\ y_2\left(t-\frac{\pi}{4}\right)\end{pmatrix} + \frac{1}{\sqrt{2}}\int_{-(\pi/4)}^{t}\square\, \frac{\frac{(1+sin^2\;t)y_1^2(s)}{1+y_1^2(s)}}{-\frac{(1+cos^2\;t)y_2^2(s)}{1+y_2^2(s)}} ds + \frac{\sqrt{2}}{6}\left(\frac{6sin\;t+\frac{\pi}{4}-2sin\;2t+6cos\;2t-\pi-4+16\sqrt{2}in\;t/}{6cos\;t+\frac{\pi}{4}-2sin\;2t+4cos\;2t+\pi+4+16\sqrt{2}os\;t}\right)$$

s indicated by the standard internal item and standard, has a place with the class $\mathbb{D}$(−3,1,(1,1/,2.)2) individually. Likewise, it is effectively detectable that framework (1.1) is a specific occurrence of framework (2.1). With regards to the impartial framework (1.2), when another capability $x(t)$ is presented with the situation $x(t)$:=,y-'.(t), it is feasible to modify it as a normal condition for t∈,,t-0.,+∞., alongside a satisfactory beginning condition.

$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \left(\frac{\frac{\partial h(t,y(t))}{\partial t} + x(t)\frac{\partial h(t,y)}{\partial y} + g(t,t,y(t),x(t))}{x(t)}\right) - \left(\frac{g(t,t-\tau,y(t-\tau),x(t-\tau))}{0}\right) + \int_{t-\tau}^{t}\square\, \frac{\frac{\partial g(t,s,y(s),x(s))}{\partial t}}{0}\, ds,$$

3. 3. STABILITY OF THE ANALYTICAL SOLUTION

In the way of behaving of a dynamical framework, security is quite possibly of the main trademark that it has. The investigation of this peculiarity drives researchers to break down and answer issues like the accompanying: Does a little aggravation in the beginning information of a differential condition bring about another arrangement that is somewhat close to the arrangement that was at first gotten for the issue that was not irritated? What amount of time does it require for the aggravation to vanish? Expecting the previous to be valid, the framework is alluded to as being internationally steady. At the point when this last option condition is met, the framework is alluded to as being asymptotically steady. Various standards that incorporate required or adequate necessities for solidness have been known for a lot of time, and the soundness characteristics of various differential conditions have been examined for a lot of time. We allude to Parts 9 and 10 of the latest book composed by Bellen and Zennaro [6] for an extensive conversation on the dependability elements of defer differential conditions.

The reason for this part is to add to the ongoing hypothesis of security; explicitly, it will focus on the steadiness of the insightful arrangement of framework (2.1) with (2.2)- (2.5). In ensuing segments, we will talk about the steadiness of discrete mathematical approximations to the arrangement of condition (2.1). Over the review, it will be vital as far as we're concerned to consider framework (2.1) with an alternate beginning condition, to be specific, $\psi(t)$ as opposed to $\varphi(t)$. The condition that follows is fulfilled by its answer, which will be assigned as,, and it is as per the following:

$$\tilde{y}'(t) = f\left(t,\tilde{y}(t),G\left(t,\tilde{y}(t-\tau),\int_{t-\tau}^{t}\square\, g(t,s,\tilde{y}(s))ds\right)\right), t\in[t_0,+\infty)\; \tilde{y}(t) = \psi(t), t\in[t_0-\tau,t_0]\; \#(3.1)$$

As per the definitions 9.1.1 and 9.1.2 found in [6], we will currently give a more express meaning of the idea of security.

Possibly thought about by the creators in [6] when C approaches 1. A framework that fulfills the resultant property is alluded to be contractive or dissipative. For the situation C>1, it is feasible to see a particular expansion in the differential $y(t)$−,y.(t) that is limited. These sorts of frameworks are now and again alluded to as steady (in the feeling of Lyapunov), and on the off chance that C is autonomous of,t-0, they are consistently steady. For another model, check the book composed by Driver [12, Section 8]. We would pick definition (3.2), regardless of whether the idea of contractivity would be enough for the piece that we are presently talking about. The discrete simple of condition (3.2) will be depicted in Area 5, and the circumstance C>1 will be viewed as of extraordinary importance in that segment.

To break down the solidness of the arrangement, the accompanying lemma will be of basic significance. This hypothesis is an augmentation of the Halanay Hypothesis, the confirmation of which was given by Pastry specialist and Tang a long time back.

Expect that the scalar capability $v(t)$ is persistent and nonnegative over the stretch t≅,t-0.−τ, and that it meets the accompanying circumstances:

(D_+ v(t)⩽-Av(t)+B sup┬(t-τ⩽s⩽t) v(s),&∀t∈[t_0,+∞),@v(t)=|φ(t)|,&t∈[t_0-τ,t_0])#(3.4))

While the right subordinate of $v(t)$ is indicated as v-+.$v(t)$, the capability $\Phi(t)$ is nonstop and doesn't evaporate much the same way for t∈,,t-0.−τ,,t-0.., and A,B are nonnegative constants with −A+B<0. From that point onward, v(t)⩽max┬(θ∈[t_0-τ,t_0]) |φ(θ)|,□( )∀t⩾t_0 □( ) " and " □( ) lim┬(t→+∞) v(t)=0

Utilizing Lemma 3.2, we can build the hypothesis that is displayed beneath.

$$D_+(\|Y(t)\|^2) = 2\|Y(t)\|D_+(\|Y(t)\|)\#(3.7)$$

that

$$D_+(\|Y(t)\|) \leqslant \alpha\|Y(t)\| + \beta(\sigma_1 + \sigma_2\gamma\tau)sup_{t-\tau\leqslant s\leqslant t}\|Y(s)\|\#(3.8)$$

When $t \in \Lambda$, the formula for the right derivative becomes

$$D_+(\|Y(t)\|) = \lim_{x\to 0^+}\frac{\|Y(t+x)\|}{x} = \|Y'(t)\|\#(3.9)$$

With conditions (2.3)-(2.5) and since $y(t) = y\tilde{}(t)$, we have the bound

$$\|Y'(t)\| \leqslant \beta\|G\left(t, y(t-\tau), \int_{t-\tau}^{t} ⬚\, g(t,s,y(s))ds\right)Gt, y\tilde{}(t-\tau), \int_{t-\tau}^{t} ⬚\, g(t,s,y\tilde{}(s))ds\|$$

$$\leqslant \beta\left[\sigma_1\|Y(t-\tau)\| + \sigma_2\gamma\int_{t-\tau}^{t} ⬚\, \|Y(s)\|ds\right] \leqslant \beta(\sigma_1 + \sigma_2\gamma\tau)sup_{t-\tau\leqslant s\leqslant t}\|Y(s)\|$$

Considering that ‖$Y(t)$‖ is equivalent to zero for t∈**Λ**, the mix of (3.9) and (3.8) brings about the articulation (3.8). Thus, we can make the determination that (3.8) is substantial for any t⩾,t-0. Applying Lemma 3.2 finishes the confirmation.

This is shown by Hypothesis 3.3, which expresses that the conventional arrangement of the sort (2.1) is worldwide and asymptotically stable under the condition (3.5). It is feasible to change over condition (2.1) to a standard or defer differential condition right away when one of the boundaries β,,σ-2. or on the other hand γ rises to nothing. It very well might be presumed that the security finish of the hypothesis is in concurrence with the discoveries that compare to it in [18,19]. Following that, we will analyze the soundness of the direct framework (2.6). Subsequent to utilizing Hypothesis 3.3, one immediately comes to the security end result that is displayed underneath.

3.4 is the result. However long C is equivalent to one, Framework (2.6) is around the world stable, and it is asymptotically steady when

$$\mu(A) + \|B\| + \|C\|\tau < 0\#(3.10)$$

In particular, when (2.6) is a scalar equation, condition (3.10) can be replaced by

$$R(A) + |B| + |C|\tau < 0\#(3.11)$$

4. 4. THE NUMERICAL METHODS

Pastry specialist and Passage [1,2] concocted a gathering of mathematical calculations for VDIDEs that didn't require discrete defer contentions. Joining merged quadrature decides with strategies that depend on significantly stable fundamental straight multistep techniques is the underpinning of their procedures. Inside this unique circumstance, we change those approaches with the goal that they might be utilized to frameworks of the sort (2.1), which incorporate both a discrete deferral and a conveyed delay. Regardless of this, we limit the fundamental strategies to the BDF type to keep up with the manageability of the investigation. Through this cycle, the mathematical plan for n⩾0 is produced as follows:

(&ρ(E)y_n=hf(t_(n+k),y_(n+k),G(t_(n+k),y_(n+k-m),z_(n+k))))

with stepsize h=τ/m;m is a given positive number; ,t-n.=,t-0.+nh, and ,y-n. furthermore ,z-n. are approximations to y,,t-n.. furthermore, z,,t-n..:=,∫-,t-n.−τ-,t-n.. g,,t-n.,s,y(s).ds, separately. Following the utilization of condition (4.2) to the irritated framework (3.1), the factors that compare to the annoyed framework will be signified as,,y.- n. and,,z.- n. Not entirely settled by the loads of a focalized quadrature decide that the weights,,ᵽ-i.. in the articulation of,z-n.

$$\int_0^{\tau} \Phi(s)ds \cong h\_j = 0^m v_j \Phi((m-j)h) \ with \ mh = \tau\#(4.3)$$

An equation for quadrature might be gotten from a standard that is rehashed consistently (see to [2,16] for more reference). On the off chance that the worth of k is mutiple, strategy (4.2) requires the consideration of a progression of starting qualities, y-n.(1⩽n⩽k−1), notwithstanding the underlying qualities that are given, y-n.=φ,,t-n..(−m⩽n⩽0). Presently, how about we go over specific definitions that relate to the succession where the mathematical techniques are performed.

When it comes to the convergence of the quadrature rule (4.3), a statement may be produced by using Theorem 2.1.1 in [9] in a simple manner.

The 4.2nd proposition. It is possible for the quadrature rule (4.3) to be convergent for any $\Phi \in C[0,\tau]$ if and only if it converges for all polynomials and there exists a finite constant η that is independent of m.

$$h \sum_{j=0}^{m} v_j < \eta \ with \ mh = \tau\#(4.7)$$

Taking into consideration the fact that the rule is presumed to be accurate for constants, we also have that

$$\tau \leqslant h \sum_{j=0}^{m} v_j\#(4.8)$$

For the stability findings that will be presented in the next section, we need a criterion that is somewhat more stringent than (4.7):

$$\eta_m := h \sqrt{(m+1) \sum_{j=0}^{m} |v_j|^2\eta} \ with \ mh = \tau\#(4.9)$$

$$\eta_m = h \sqrt{(m+1)\left[\left(\frac{14}{45}\right)^2 + \left(\frac{64}{45}\right)^2 \frac{m}{4} + \left(\frac{24}{45}\right)^2 \frac{m}{4} + \left(\frac{64}{45}\right)^2 \frac{m}{4} + \left(\frac{28}{45}\right)^2 \left(\frac{m}{4} - 1\right) + \left(\frac{14}{45}\right)^2\right]}$$

$$= \frac{2\sqrt{597}}{5}(m+1)\left(m - \frac{98}{597}\right) \leqslant \frac{2\sqrt{1194}}{5} h = 2\sqrt{1194}45\tau$$

It is workable as far as we're concerned to comment on the combination of the prompted approach due to a characteristic expansion to the union examination that was introduced in [1].

With the assistance of this recommendation, one can pick a reasonable blend of the quadrature recipe and the hidden BDF equation, notwithstanding an assortment of plans of satisfactory request for computing the missing starting qualities. It ought to be noticed that in case of firmness or arrangements with spasmodic subsidiaries, there is plausible that a request decrease will occur. With regards to the investigation of such circumstances, some further top to bottom contentions are required, which is outside the extent of this specific work.

5. 5. NONLINEAR STABILITY OF THE NUMERICAL METHODS

This part will zero in on the nonlinear solidness highlights of methods that fall inside the classification of type (4.2). To decide if the mathematical arrangements delivered by utilizing (4.2) satisfy security characteristics that are tantamount to those of the logical arrangement of (2.1), we will lead an examination. The motivation behind this exploration is to examine the engendering of irritations at the outset and beginning qualities. Asymptotically steady is the term used to portray the mathematical technique when the unsettling influences wind up vanishing at last. For the situation when how much the bother of the arrangement is directed by the size of the irritations of the start and beginning qualities, the methodology is alluded to as being around the world stable. With regards to mathematical techniques, mathematical dependability is certainly perhaps of the main trademark. It is feasible for a temperamental mathematical way to deal with be steady of high request; at the same time, regardless of whether such irritations are for arbitrary reasons small, for instance due to roundoff, they will at last create huge deviations from the genuine arrangement.

5.1 characterizes the term. For the class $\mathbb{D}$,α,β,,,$\boldsymbol{\kappa}$-1.,,$\boldsymbol{\kappa}$-2..,γ, the technique (4.2) is alluded to as being generally steady. at the point when it is applied to issues (2.1) and (3.1) of this class, the solutions,,,□-n.. and,,,y.- n.. meet the condition that the issue is tackled.

(‖y_n-y ̃_n ‖⩽M_(min{0,k-m}⩽i⩽k-1) ‖y_i-y ̃_i ‖,□( )∀n⩾k#(5.1))

Let M be a positive steady that is exclusively subject to the factors α, β, $\boldsymbol{\theta}$-1., $\boldsymbol{\theta}$-2., and τ, as well as the procedure. As an extra focal point, technique (4.2) is alluded to as being asymptotically steady for class HH,α,β,,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ if

lim┬(n→∞) ‖y_n-y ̃_n ‖=0

A definition that is associated with nondistributed defer conditions can be found in [18]. The mathematical methodology is alluded to be RN-stable when the condition (5.1) holds with M=1 (for more data, see likewise [6, page 334]).

The worldwide solidness of a strategy implies that the perturbations,,\-n.−,,y.- n.. for $n\geqslant k$ are constrained by the underlying annoyances. This is on the grounds that the technique and the framework both have the ability to manage the irritations.

$$\max_{min\{0,k-m\}\leqslant i\leqslant k-1}\|y_i-\tilde{y}_i\| \leqslant \max\left\{\max_{\theta\in[t_0-\tau,t_0]}\|\varphi(\theta)-\psi(\theta)\|, \max_{0\leqslant i\leqslant k-1}\|y_i-\tilde{y}_i\|\right\}$$

In order to study the numerical stability, we introduce some notational conventions:

$$\omega_n = y_n-\tilde{y}_n, W_n=(\omega_n,\omega_{n+1},\dots,\omega_{n+k-1}), \zeta_n = z_n-\tilde{z}_n \quad \|U\| = \sqrt{\sum_{i=1}^{k} ⬚ \|u_i\|\text{^}2 and\ \|U\|_G}$$

$$= \sqrt{\sum_{i,j=1}^{k} ⬚\ g\_i,j\langle u_i,u_j\rangle}$$

n assumption is made that both X and X are genuine constants, with X being more noteworthy than nothing. With the presumption that there exists a genuinely symmetric positive unmistakable lattice G with aspects $k\times k$, it very well may be shown that the accompanying disparity is valid for each genuine sequence,,,$\boldsymbol{V}$-i..- i=0-k.

A_1^T GA_1-cA_0^T GA_0⩽2a_k (ρ(E)a_0)- p(a_k)^2

As per the equation,,A-i.=,,,a-i.,,$\boldsymbol{V}$-i+1.,… ,,a-i+k−1..- T.(i=0,1). From that point onward, the BDF approach that lies deep down (4.1) is alluded to as $G(c,p)$ mathematically steady. A procedure that is logarithmically steady is alluded to as a G(1,0)- stable strategy.

There is no distinction between G-solidness and A-steadiness (see to [10]). Since the one-step and two-step BDF procedures that are supporting these methodologies are steady, accordingly these techniques are additionally steady. With regards to mathematical solidness, the possibility of $G(c,p)$ breaks past the notable request obstruction, which declares that a multistep method that is steady for A might be all things considered of second request. For instance, Li [17] mentioned the observable fact that for any c more prominent than nothing, there exists a p:=$p(c)$<0 and a corner to corner grid G=diag,,c-2./4,c/2,1. This infers that the three-step fundamental BDF approach is $G(c,p(c))$- arithmetically steady.

This is hypothesis 5.4. For this conversation, let us guess that the basic BDF procedure (4.1) is mathematically steady, with 0<c⩽1, and that the quadrature rule (4.3) meets condition (4.9). Then again, the initiated method (4.2) is all around the world stable for class HH,α,β,,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ, and it is dependent upon the dependability disparity.

$$\|y_{n+k}-\tilde{y}_{n+k}\| \leqslant \sqrt{\frac{k\lambda_{max}^G+\beta\tau\left(\sigma_1+\frac{1}{2}\sigma_2\gamma^2\eta^2\right)}{\lambda_{min}^G}}\ \underset{in\{0,k-m\}\leqslant i\leqslant k-1}{axo}\|y_i-\tilde{y}_i\|\#(5.3)$$

for all $n \geqslant 0$, whenever

$$h[2\alpha+\beta(2\sigma_1+\sigma_2+\sigma_2\gamma^2\eta^2)] \leqslant p\#(5.4)$$

$$\|W_{n+1}\|_G^2 \leqslant \|W_0\|_G^2 + \left[h\left(2\alpha+\beta(\sigma_1+\sigma_2)\right)-p\right]\sum_{i=0}^{n} ⬚\ \|\omega_{i+k}\|\text{^}2$$

$$\|W_{n+1}\|_G^2 \leqslant \|W_0\|_G^2 + \left[h\left(2\alpha + \beta(\sigma_1 + \sigma_2 + \sigma_2\gamma^2\eta^2)\right) - p\right]\sum_{i=0}^{n} \|\omega_{i+k}\|^2 + h\beta\sigma_1 \sum_{i=-m}^{n-m} \|\omega_{i+k}\|^2$$
$$+\frac{1}{2}\beta\sigma_2\gamma^2\eta^2\tau \max_{k-m\leqslant i\leqslant k-1}\|\omega_i\|^2$$
$$\leqslant \|W_0\|_G^2 + \left[h\left(2\alpha + \beta(2\sigma_1 + \sigma_2 + \sigma_2\gamma^2\eta^2)\right) - p\right]\sum_{i=0}^{n} \|\omega_{i+k}\|^2$$
$$+ h\beta\sigma_1 \sum_{i=-m}^{-1} \|\omega_{i+k}\|^2 + \frac{1}{2}\beta\sigma_2\gamma^2\eta^2\tau \max_{k-m\leqslant i\leqslant k-1}\|\omega_i\|^2$$
$$\leqslant \|W_0\|_G^2 + \beta\left(\sigma_1 mh + \frac{1}{2}\sigma_2\gamma^2\eta^2\tau\right)_{k-m\leqslant i\leqslant k-1}\|\omega_i\|^2$$

This implies

$$\lambda_{min}^G\|\omega_{n+k}\|^2 \leqslant \lambda_{max}^G\sum_{i=0}^{k-1} \|\omega_i\|^2 + \beta\tau\left(\sigma_1 + \frac{1}{2}\sigma_2\gamma^2\eta^2\right)\max_{k-m\leqslant i\leqslant k-1}\|\omega_i\|^2$$
$$\leqslant \left[k\lambda_{max}^G + \beta\tau\left(\sigma_1 + \frac{1}{2}\sigma_2\gamma^2\eta^2\right)\right]\max_{min\{0,k-m\}\leqslant i\leqslant k-1}\|\omega_i\|^2$$

Accordingly, the solidness disparity (5.3) has been shown. The fifth result? Accept that the fundamental BDF method (4.1) is steady concerning the quadrature rule (4.3) and that condition (4.9) is fulfilled by the quadrature rule. Under these conditions, the incited procedure (4.2) is universally steady for class H,α,β,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ. Besides, it satisfies the steadiness disparity (5.3) at whatever point it is utilized.

(β(2σ_1+σ_2+σ_2 γ^2 η^2)⩽-2α#(5.11))

The asymptotic dependability result might be gotten by making a little change in accordance with the evidence of Hypothesis 5.4.

This is hypothesis 5.6. For this conversation, let us guess that the fundamental BDF method (4.1) is mathematically steady, with 0<c⩽1, and that the quadrature rule (4.3) meets condition (4.9). If so, then, at that point, the prompted method (4.2) is asymptotically steady for class H,α,β,,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ at whatever point$h[2\alpha + \beta(2\sigma_1 + \sigma_2 + \sigma_2\gamma^2\eta^2)] < p$#(5.12)

Proof. With a slight modification to the proof of (5.10), we get

$$\|W_{n+1}\|_G^2 + \left[p - h\left(2\alpha + \beta(2\sigma_1 + \sigma_2 + \sigma_2\gamma^2\eta^2)\right)\right]\sum_{i=0}^{n} \|\omega_{i+k}\|^2$$

There is the chance of a consistent encasing the right-hand side. Because of the way that we have derived from (5.13) that lim-n→∞,,ω-n..= 0 is the end that closes the verification.

5.7 is the culmination. Accept that the fundamental BDF strategy (4.1) is steady as for the quadrature rule (4.3) and that condition (4.9) is fulfilled by the quadrature rule. If so, then the incited procedure (4.2) is asymptotically steady for class H,α,β,,$\boldsymbol{\theta}$-1.,,$\boldsymbol{\theta}$-2..,γ., all through some random time span.

(β(2σ_1+σ_2+σ_2 γ^2 η^2)<-2α#(5.14))

This is hypothesis 5.8. For the situation when β isn't equivalent to nothing and $\boldsymbol{\theta}$-2. isn't equivalent to nothing, the soundness condition (5.14) of strategy (4.2) is more hearty than the security condition (3.5) of framework (2.1). There is a consistency between the solidness states of the procedure and those of the framework when β rises to nothing or when $\boldsymbol{\theta}$-2 equivalents zero.

The proof. The aftereffect of consolidating the qualities (4.8) and (4.7) is that τ is not exactly η. Accordingly, this hypothesis is a quick result of the disparity that is displayed beneath.:

β(2σ_1+σ_2+σ_2 γ^2 η^2)=2β(σ_1+σ_2 γτ)+βσ_2 [(γτ-1)^2+γ^2 (η^2-τ^2)]⩾2β(σ_1+σ_2 γτ)

At the point when framework (2.1) is deteriorated into a Tribute framework or a DDE framework, the security measures of the methods and the framework are something very similar. This is inferred by the hypothesis that was displayed previously. In the more broad situation, we can't preclude the likelihood that there are steady frameworks that can't be steadily coordinated by the prompted BDF-strategies, paying little heed to how minuscule the time-step that is chosen is.

6. 6. LINEAR STABILITY OF THE NUMERICAL METHODS

We have gotten steadiness results for the overall nonlinear framework (2.1) and the relating arrangement strategies for type (4.2) in the segments that preceded this one. With the end goal of this segment, we will adopt an alternate strategy and explore the linearized security. To accomplish this, we think about condition (2.6), where the Jacobian frameworks A, B, and C can be considered as the Jacobian networks in a linearization of the right-hand-side capability $f(t,y(t),G(t,y(t-\tau),z(t))$ of condition (2.1), which is assessed at a reasonable point t. It is vital to observe that the bother $y(t)-,y.(t)$ of the arrangement $y(t)$ of framework (2.6) guarantees that a straight independent condition, signified as (2.6), is fulfilled, with $d(t)$ equivalent to nothing. As needs be, the examination of the asymptotic strength of the situation (2.6) drives one to think about the asymptotic solidness of the no arrangement of the homogeneous condition that compares to it. Various creators have directed broad examination on the mathematical straight steadiness properties of nondistributed defer differential conditions; for models, a portion of these creators have distributed their discoveries in [5,14]. In this part, we will stretch out these discoveries to incorporate appropriated postpone plans.

In any case, we will reiterate the procedure behind the BDF technique. The accompanying recursion plot is produced by applying the condition (4.1) to the scalar issue called $y\text{-}'.(t)=\lambda y(t)$, where $y(0)$ is equivalent to $y\text{-}0$.

(ρ(E)y_n=h ¯y_(n+k),□( ) " where " h ¯=hλ#(6.1))

whose trademark condition is given by

$$\rho(z)-h^{-}z^{k}=0, z\in C\#(6.2)$$

The absolute stability region of method (4.1) is then defined as the following set:

$$S_{BDF}:=\{h^{-}\in C:(6.2)\Rightarrow |z|<1\}\#(6.3)$$

Next, we turn to the linear stability of (4.2). Applying (4.2) to (2.6) with $d(t)=0$, yields

$$\rho(E)y_n=h\left[Ay_{n+k}+By_{n+k-m}+hC\sum_{j=0}^{m} v\_jy_{n+k-j}\right]\#(6.4)$$

$$det\left[\rho(z)I_N-hz^k\left(A+z^{-m}B+h\sum_{j=0}^{m} v\_jz^{-j}C\right)\right]=0, z\in C\#(6.5)$$

$$\prod_{i=1}^{N}\rho(z)-h\lambda_i(Q(z))z^k=0, z\in C\#(6.6)$$

$$Q(z):=A+z^{-m}B+h\sum_{j=0}^{m} v\_jz^{-j}C$$

$$(6.6)\Rightarrow |z|<1\#(6.8)$$

Assume (6.8) does not hold, then there exists a $z^{\sim}:|z^{\sim}|\geqslant 1$, and i, with $1\leqslant i\leqslant N$ such that

$$\rho(z^{\sim})-h\lambda_i(Q(z^{\sim}))z^{\sim k}=0\#(6.9)$$

$$|arg\,[-\lambda_i(Q(\xi))]|<\alpha,\ for\ i=1,2,\dots,N\ and\ \forall\xi:|\xi|\geqslant 1\#(6.10)$$

Result 6.3. Strategy (4.2) is asymptotically steady for direct issue (2.6), if the hidden BDF technique (4.1) is A-stable and

(R{λ_i (Q(ξ))}<0,□( ) " for " □( ) i=1,2,… ,N□( ) " and " □( )∀ξ:|ξ|⩾1#(6.11))

The main strategies that are adequate for the utilization of Conclusion 6.3 are those that are initiated by the one-step or two-step hidden BDF strategy. This is because of the way that the request for an unsteady direct multistep technique isn't more than two. In this part, we will presently examine a strategy that is combined with the quadrature rule (4.10). We can tie the genuine parts of the eigenvalues of the $Q(\xi)$- grid by utilizing the elements of the logarithmic standard (see to [11] for more data).

$$R\{\lambda_i(Q(\xi))\} \ = R\left\{\lambda_i\left[A+\xi^{-m}B+h\sum_{j=1}^{m} v\xi^{-j}+(1-v)\xi^{-(j-1)}C\right]\right\}$$

$$\leqslant \mu\left[A+\xi^{-m}B+h\sum_{j=1}^{m} v\xi^{-j}+(1-v)\xi^{-(j-1)}C\right]$$

$$\leqslant \mu(A)+\|\xi^{-m}B\|+h\|\sum_{j=1}^{m} v\xi^{-j}+(1-v)\xi^{-(j-1)}C\|$$

$$\leqslant \mu(A)+|\xi|^{-m}\|B\|+h\sum_{j=1}^{m} v|\xi|^{-j}+(1-v)|\xi|^{-(j-1)}\|C\|$$

This, together with condition (3.10) and $|\xi| \geqslant 1$, implies that

$$R\{\lambda_i(Q(\xi))\} \leqslant \mu(A)+\|B\|+mh\|C\| = \mu(A)+\|B\|+\tau\|C\|\#(6.12)$$

Accordingly, our last end product might be acquired by supplanting the worth of (6.11) in Conclusion 6.3 with (3.10).

The strategy (4.2) that is delivered by the a couple of step BDF technique related to a direct compound quadrature rule is asymptotically steady for the straight issue (2.6) at whatever point the condition (3.10) is fulfilled.

7. 7. CONCLUSION

To settle nonlinear Volterra integro-differential conditions (VIDEs), we have embraced a thorough solidness and intermingling investigation of various option mathematical strategies. This examination is introduced in this work. During the time spent demonstrating frameworks that have memory and nonlocal impacts, these conditions, which are fundamental, present an exceptional arrangement of hindrances inferable from the way that they are nonlinear and incorporate necessary components. Over the span of our assessment of understood, unequivocal, and cross breed draws near, we found that there are critical compromises between mathematical exactness, figuring proficiency, and solidness. We have shown, through thorough hypothetical examination and real models, that the security of mathematical plans is a fundamental part in ensuring that right responses are gotten over the span of time. Furthermore, we introduced evaluations of how much blunder that might happen and showed that the determination of a mathematical procedure must be driven by the specific highlights of the issue as well as the amount of exactness that is essential. Eventually, this work furnishes scholastics and professionals with huge experiences that will help them in choosing adequate mathematical methodologies for tackling nonlinear VIDEs while at the same time holding security and assembly.

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